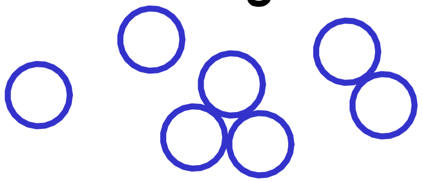
EXAMPLES OF USING THE PROBABILISTIC METHOD

CONTENTS:

1) covering points with disks
2) Ramsey numbers
3) tournaments
4) large bipartite subgraphs
5) large independent sets
6) large dominating sets
7) 2-coloring n-uniform hypergraphs

Game: • player 2 has an infinite number of unit-radius coins that may be placed anywhere in the plane without overlap



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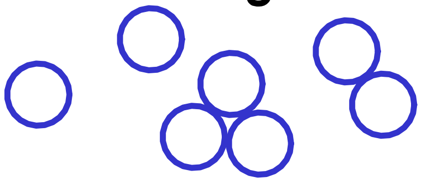
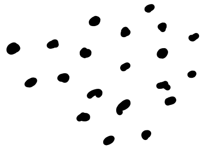


1) player 1 knows the coin size and can draw k points in the plane



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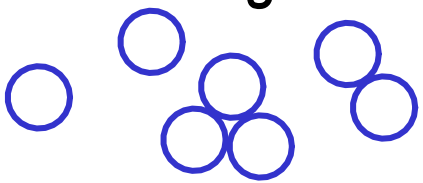

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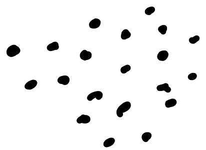

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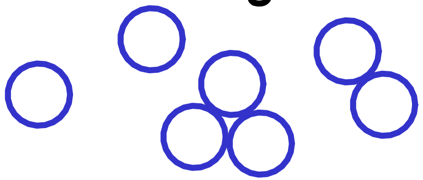

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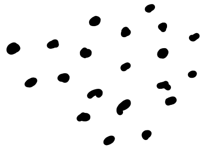

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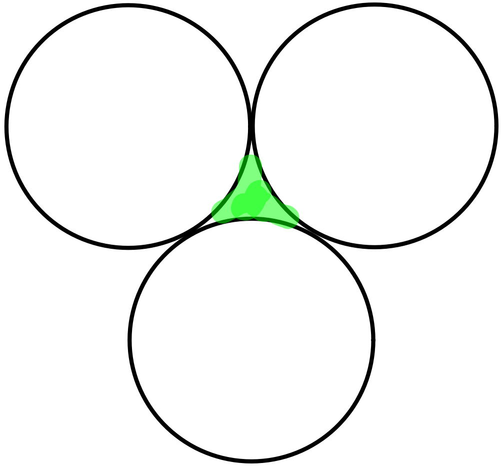
For what k does each player win?

Large k: points can't be covered

Just pack densely



Unavoidable gap

45 points suffice for player 1 to win

Suppose $k=10$. Player 1 gets to place 10 points

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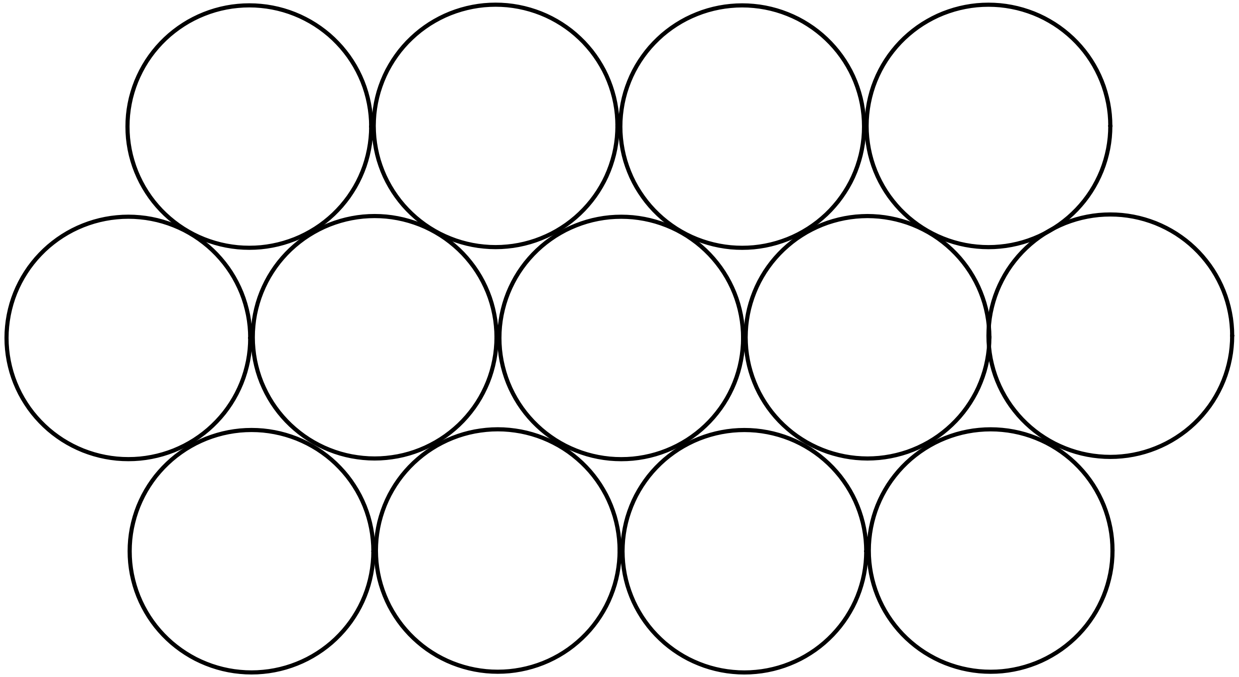
Player 2 strategy: place ∞ coins in packed formation



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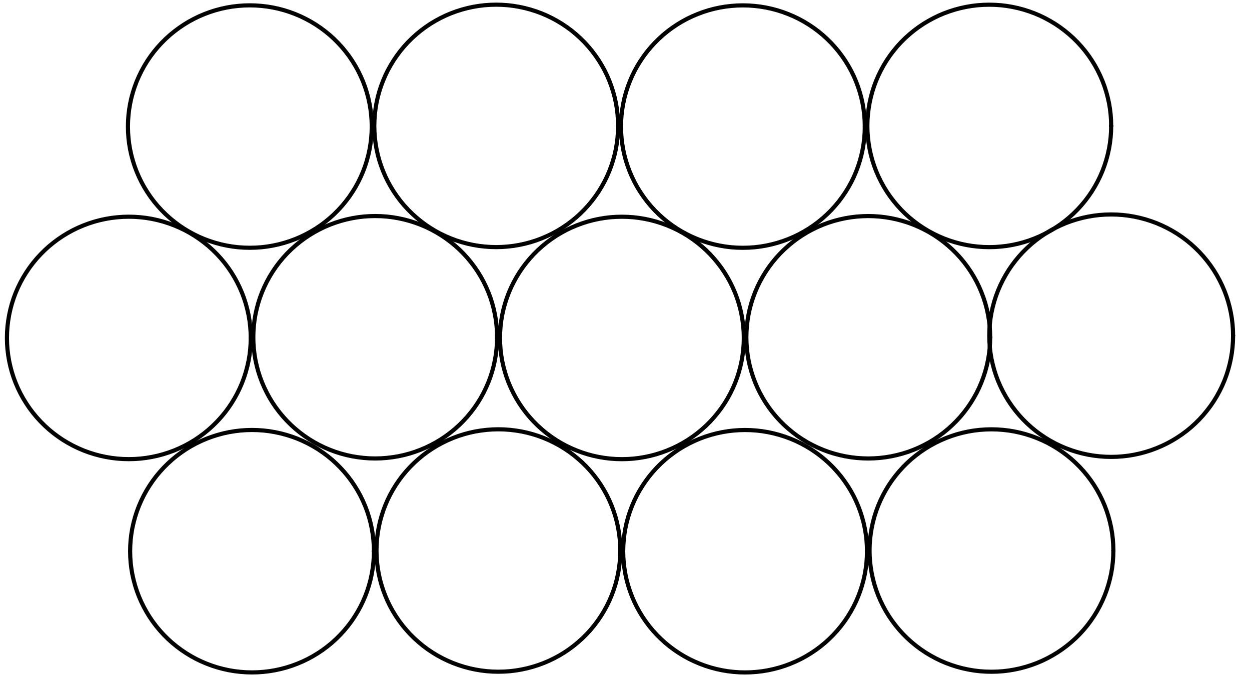
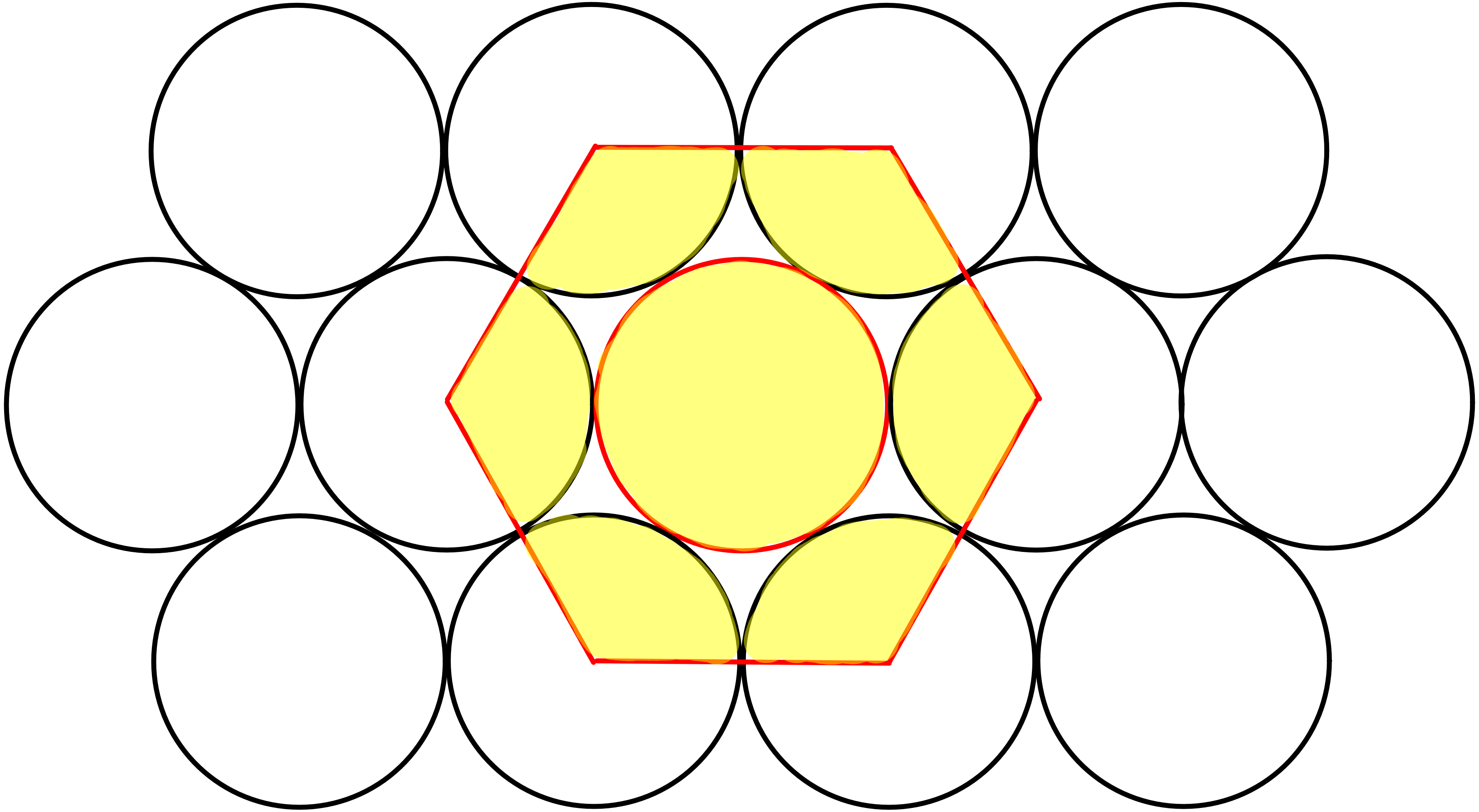


Claim: some shift of this pattern will cover all points!

$\rightarrow$ ≤ 10 coins actually required

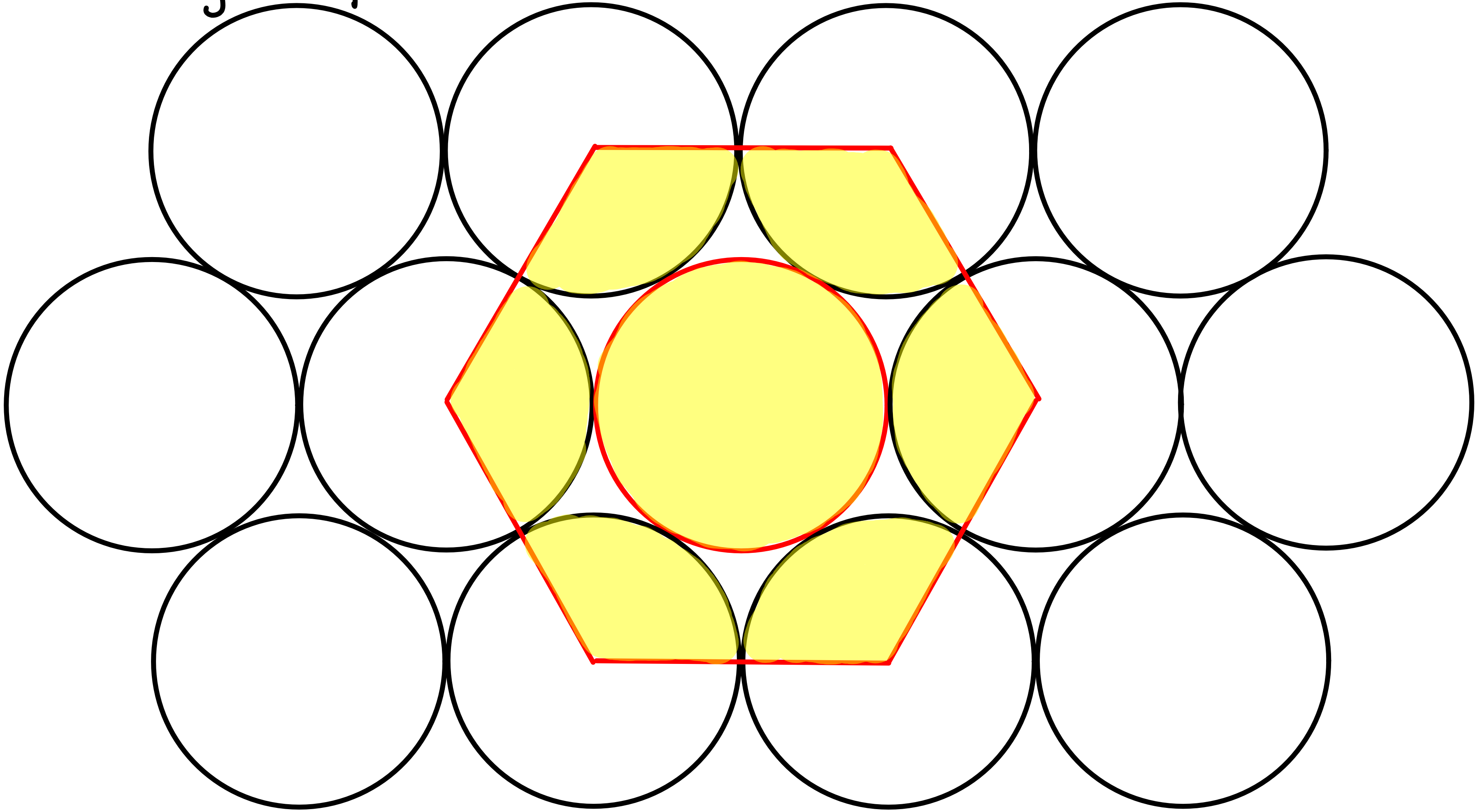
Hexagon tiles the plane.



Hexagon tiles the plane.

Hexagon density ~ 0.9069

↳ coverage of plane



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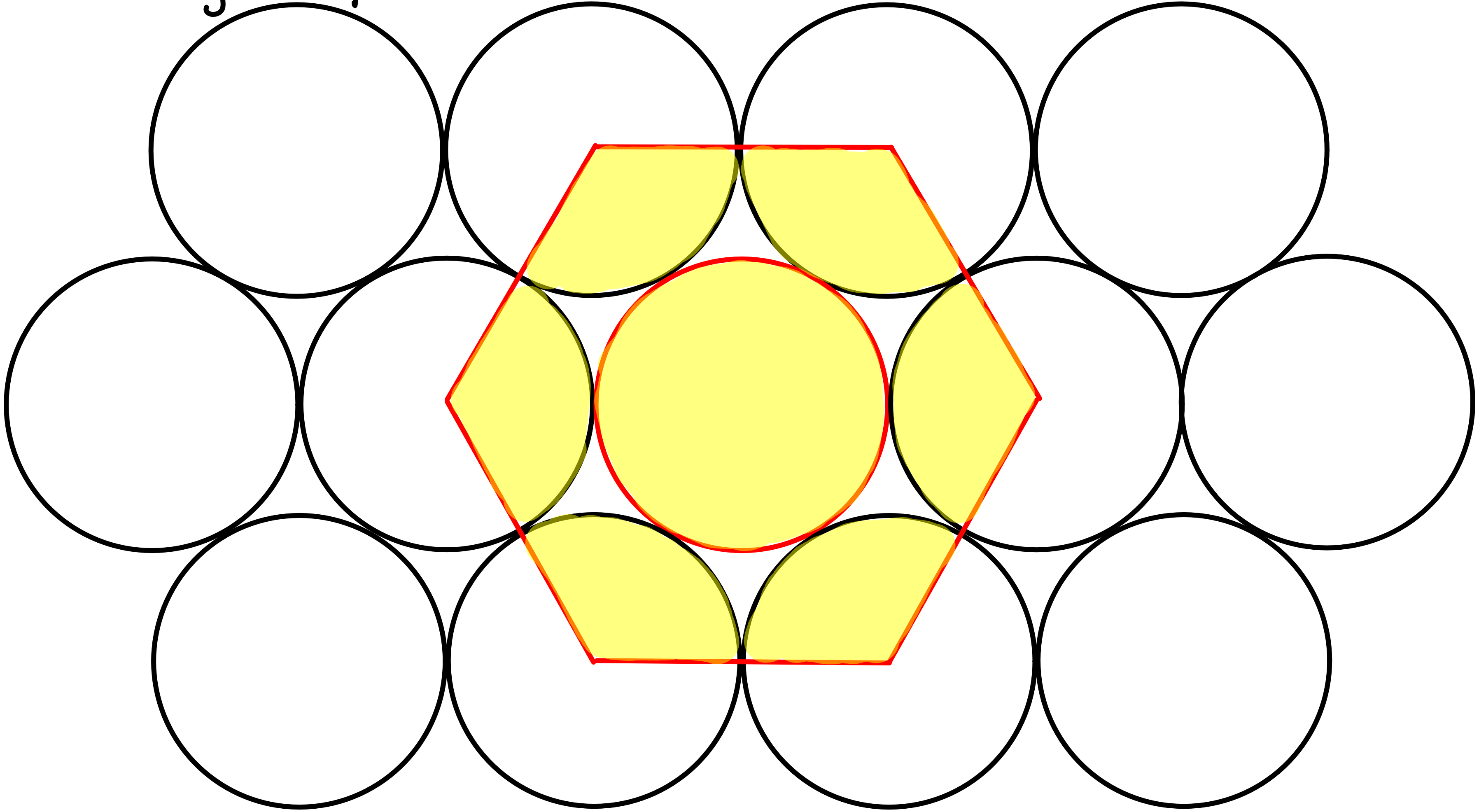

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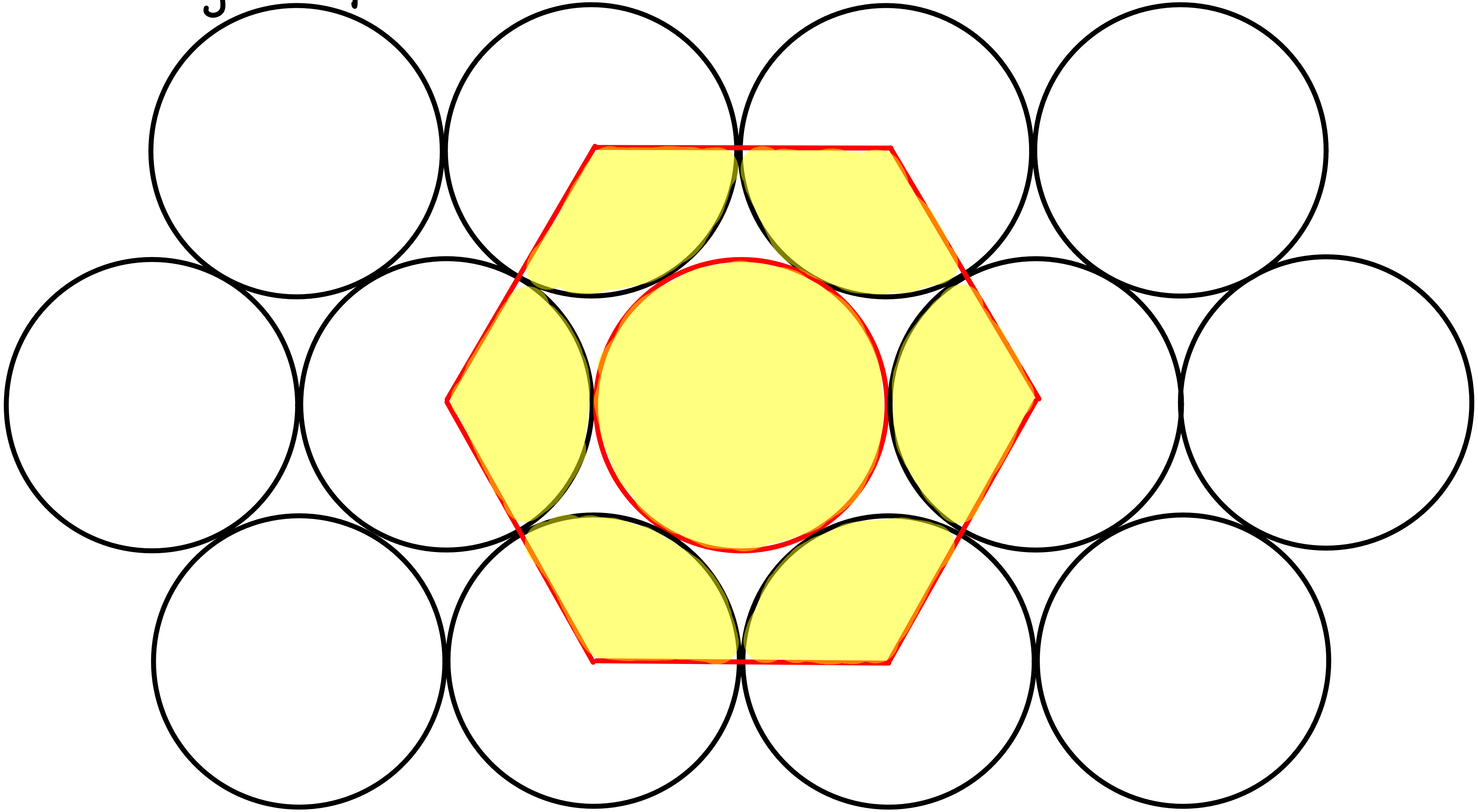

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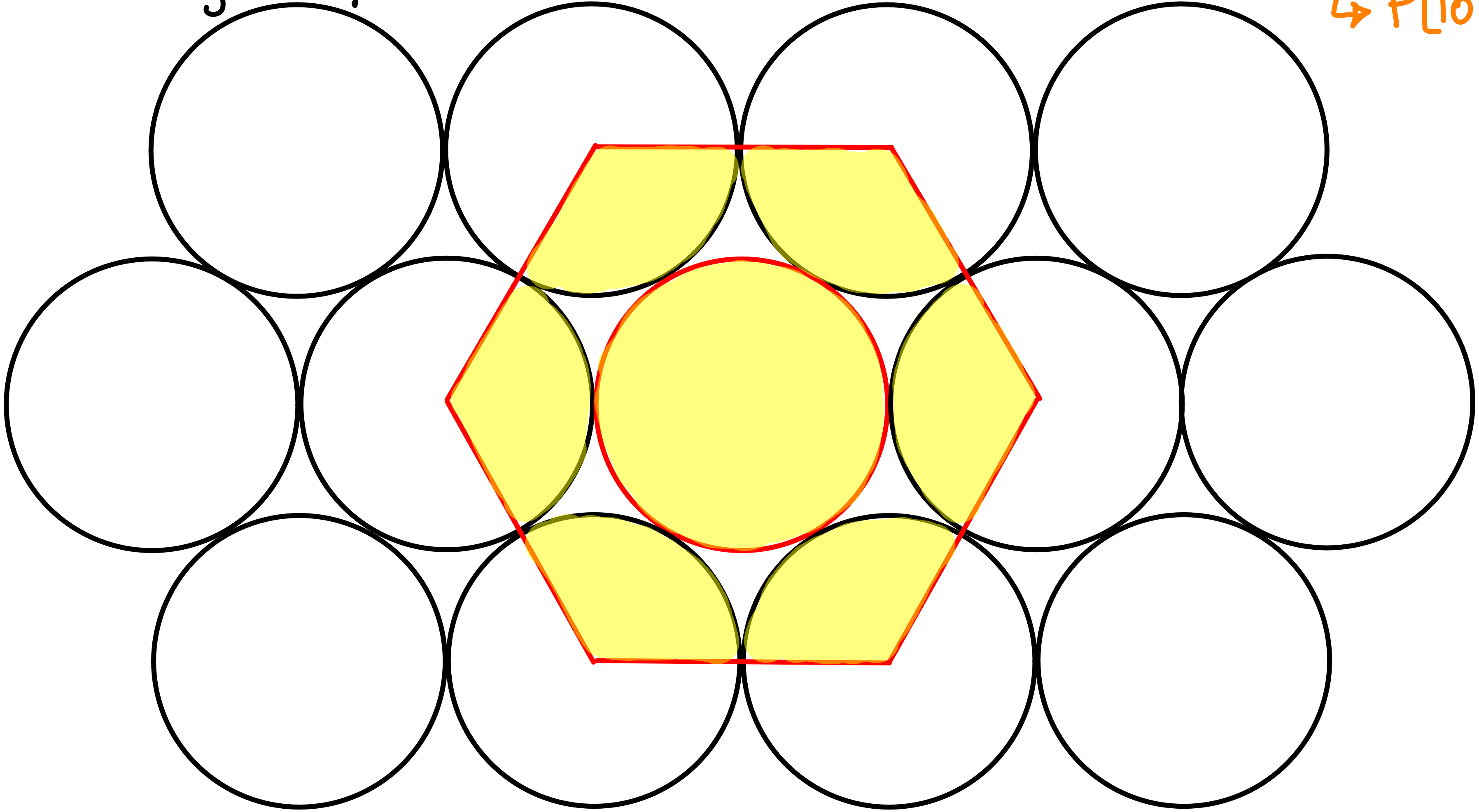

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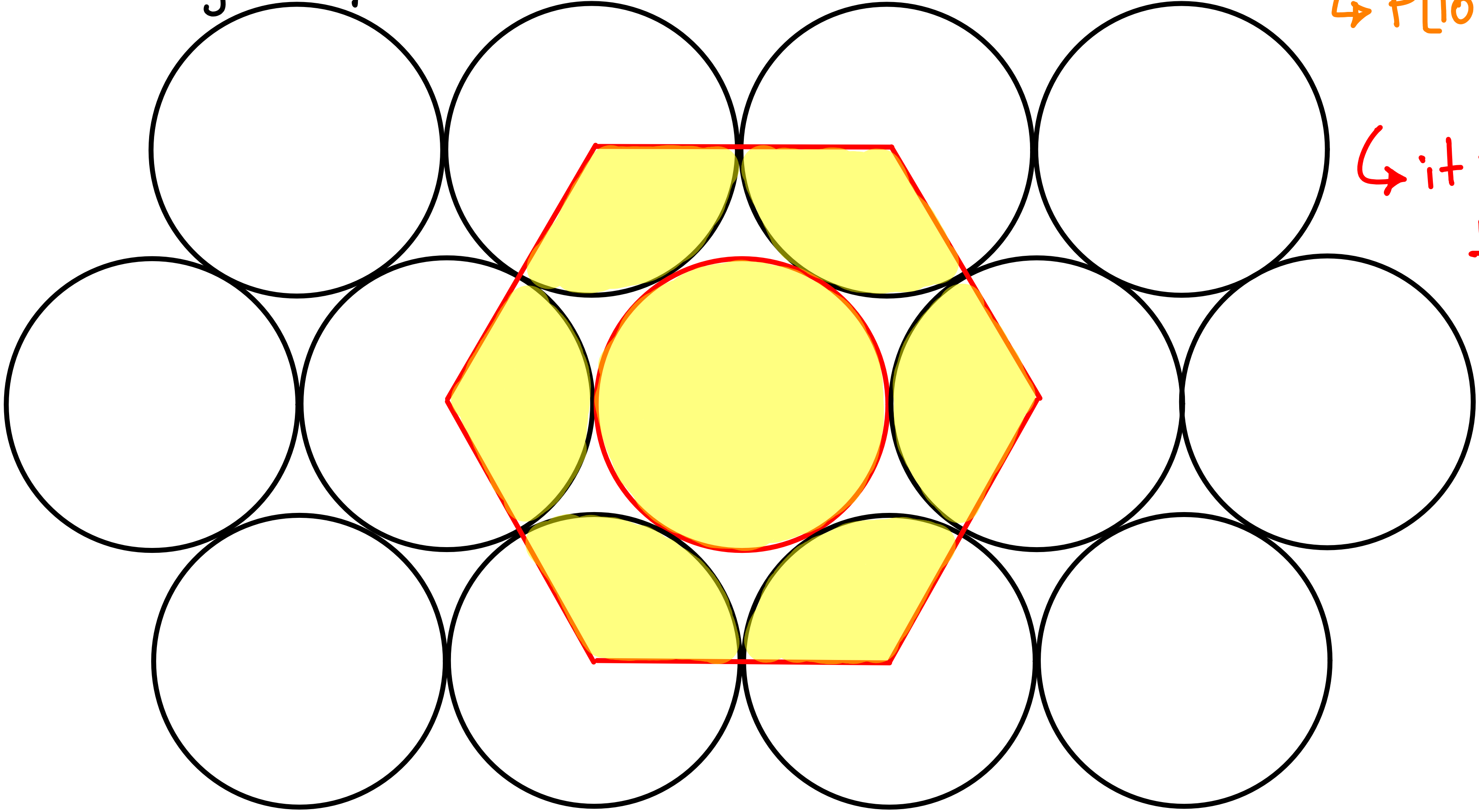
↳ P[10 points are covered] > 0

↳ it is always possible to cover all 10 !



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Every point gets covered with probability > 0.9

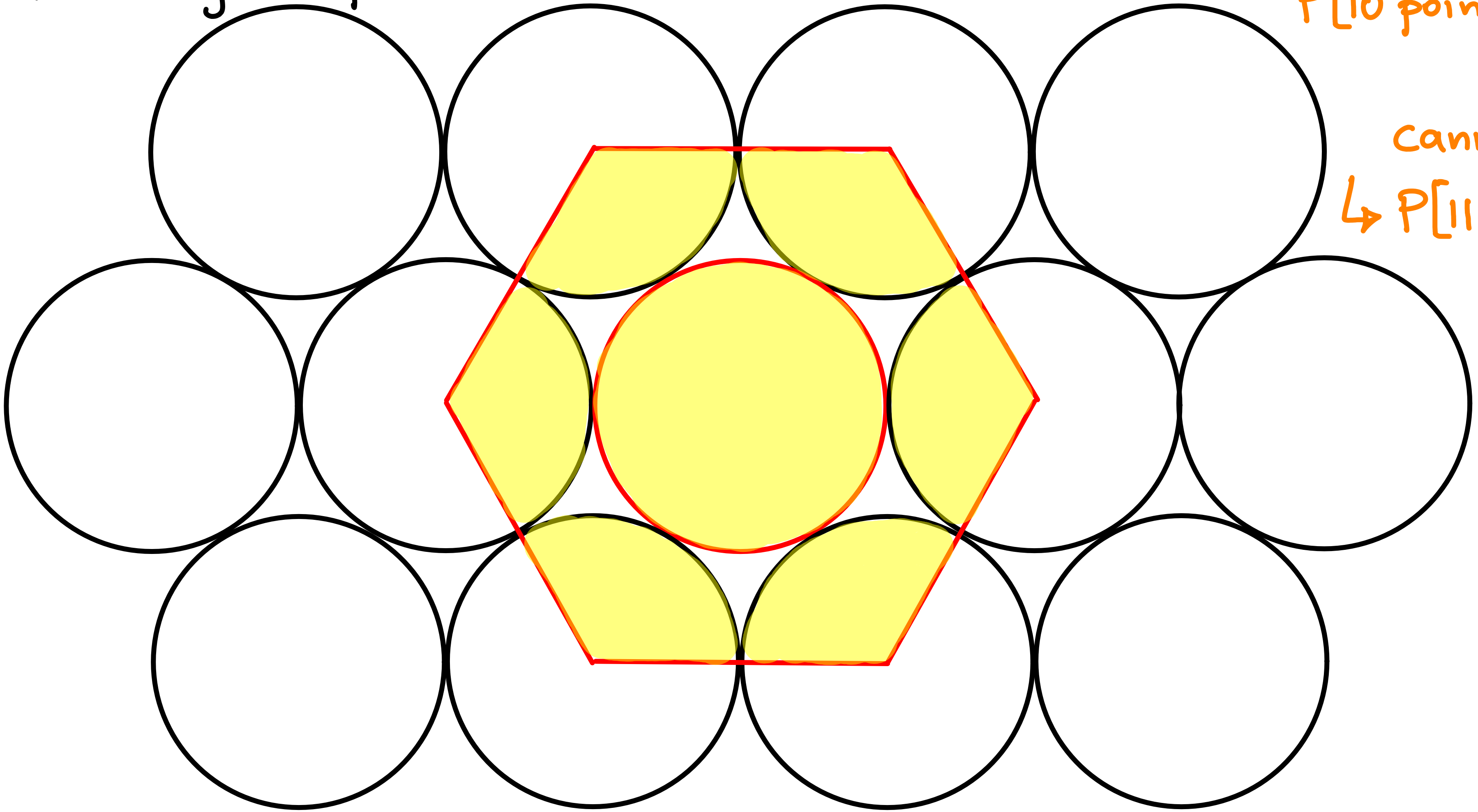
For 11 points, E[covered pts] > 0.9069·11 ≈ 9.98

P[10 points are covered] > 0

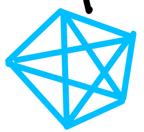
but

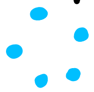
cannot conclude

↳ P[11 points are covered] > 0



Given $c \geq 3$, how large must n be so that every graph with n vertices contains a clique of size c or an independent set of size c?





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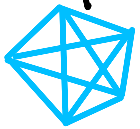

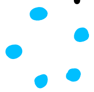

Equivalent: how large must n be so that for every 2-coloring of the edges of K_n there is a monochromatic K_c?

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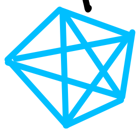
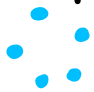
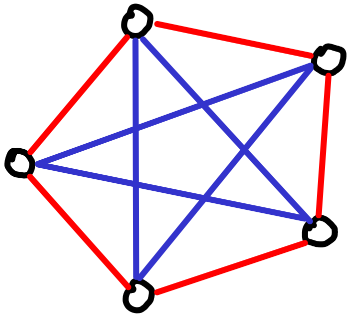




Equivalent: how large must n be so that for every 2-coloring of the edges of K_n there is a monochromatic K_c?

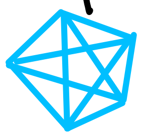
↪ color every edge red or blue → creates a red K_c or a blue K_c

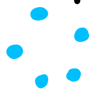
e.g.: $c = 3$



$\Rightarrow n > 5$

Given $c \geq 3$, how large must n be so that every graph with n vertices contains a clique of size c or an independent set of size c?





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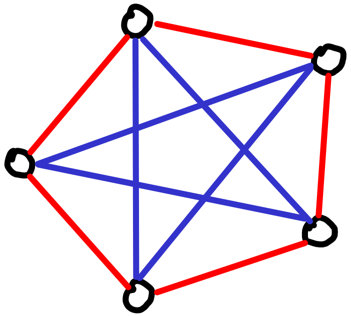

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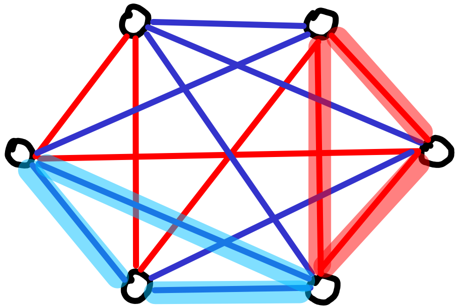

?

c-cliques in $K_n = \binom{n}{c}$

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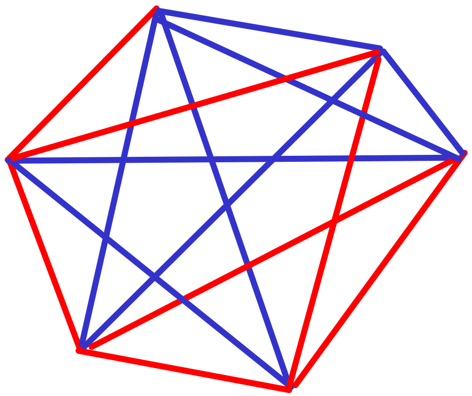

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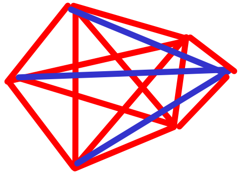

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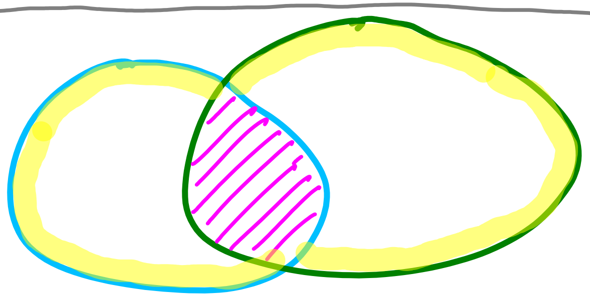

$P(A \vee B) = P(A) + P(B) - P(A \cup B)$

$\rightarrow \quad P(A \vee B) \leq P(A) + P(B)$

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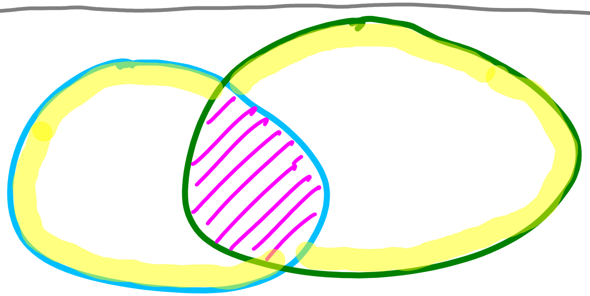

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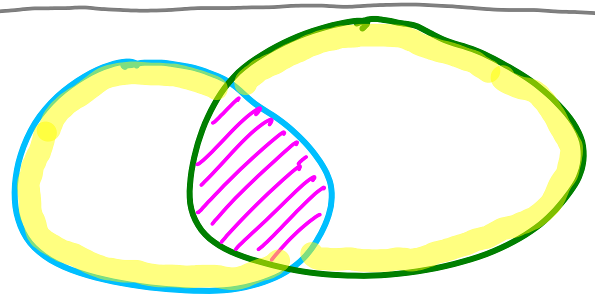

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$P(\geq 1 \text{ monochromatic } c\text{-clique}) = \binom{n}{c} \cdot 2 \cdot \left(\frac{1}{2}\right)^{\binom{c}{2}}$

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$$\binom{n}{c} \cdot 2 \cdot \left(\frac{1}{2}\right)^{\binom{c}{2}} = 2 \cdot \frac{n!}{c!(n-c)!} \cdot \left(\frac{1}{2}\right)^{\frac{c\cdot(c-1)}{2}} = 2 \cdot \frac{n!}{c!(n-c)!} \cdot \left(\frac{1}{2}\right)^{\frac{c^2-c}{2}}$$

$$< 2 \cdot \frac{n^c}{c!} \cdot \left(\frac{1}{2}\right)^{\frac{c^2}{2}} \cdot 2^{c/2} = \frac{2^{1+c/2}}{c!} \cdot \frac{n^c}{2^{\frac{1}{2}c^2}}$$

$$\boxed{= \frac{2^{1+c/2}}{c!} \cdot \frac{2^{\frac{1}{2}c^2}}{2^{\frac{1}{2}c^2}} < 1}$$

if we set $n = 2^{c/2}$

$P(\geq 1 \text{ monochromatic } c\text{-clique}) = \binom{n}{c} \cdot 2 \cdot \left(\frac{1}{2}\right)^{\binom{c}{2}}$

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$$\boxed{= \frac{2^{1+c/2}}{c!} \cdot \frac{2^{\frac{1}{2}c^2}}{2^{\frac{1}{2}c^2}} < 1 \quad \text{if we set } n = 2^{c/2}}$$

So if $n \leq 2^{c/2}$, $\exists$ graph without monochromatic K_c

$P(\geq 1 \text{ monochromatic } c\text{-clique}) = \binom{n}{c} \cdot 2 \cdot \left(\frac{1}{2}\right)^{\binom{c}{2}}$

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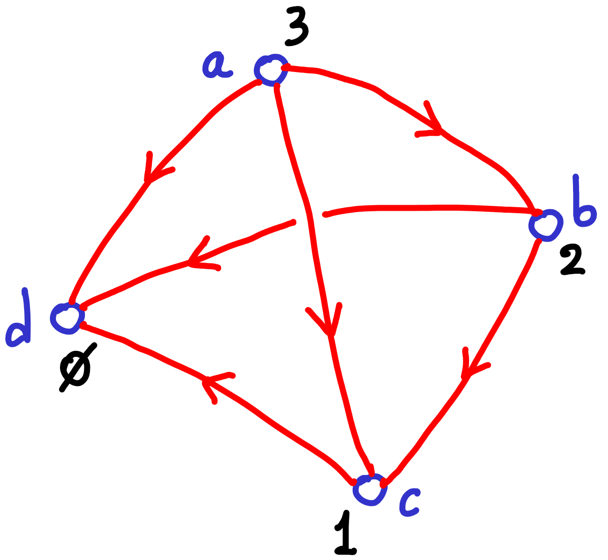
$$\boxed{= \frac{2^{1+c/2}}{c!} \cdot \frac{2^{\frac{1}{2}c^2}}{2^{\frac{1}{2}c^2}} < 1 \quad \text{if we set } n = 2^{c/2}}$$

So if $n \leq 2^{c/2}$, $\exists$ graph without monochromatic K_c

We need $n > 2^{c/2}$ to hope for guaranteed c-clique // certain if $n < \binom{2c-2}{c-1} = O(4^c)$

TOURNAMENTS

n players, everybody plays against everybody
(vertices) (directed edge $\overrightarrow{xy}$: x beat y)



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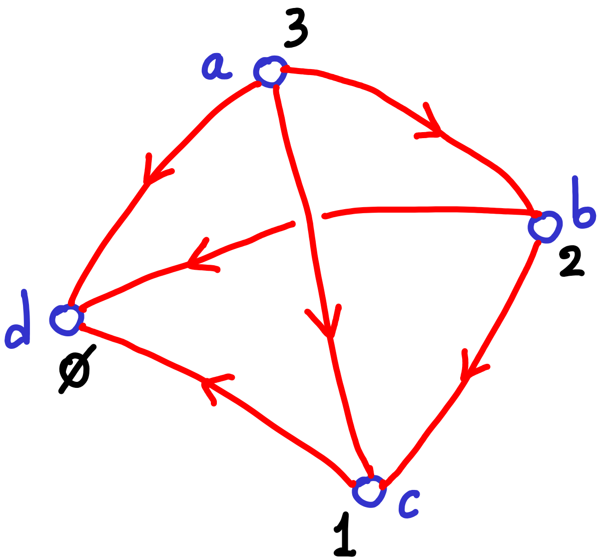

Wiki :

"many of the important properties of tournaments were first investigated... in order to model

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TOURNAMENTS

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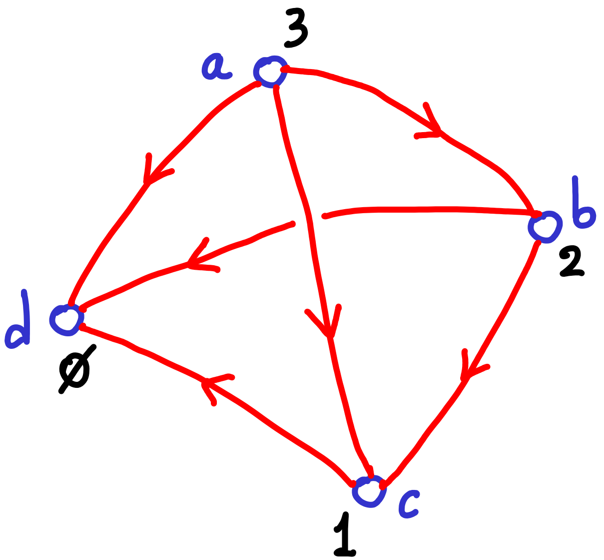

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TOURNAMENTS

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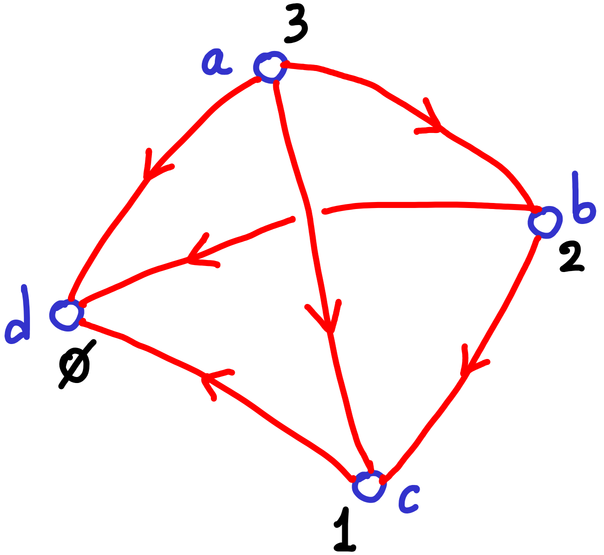

Wiki:

"many of the important properties of tournaments were first investigated... in order to model dominance relations in flocks of chickens"

perhaps more interesting: voting theory

TOURNAMENTS

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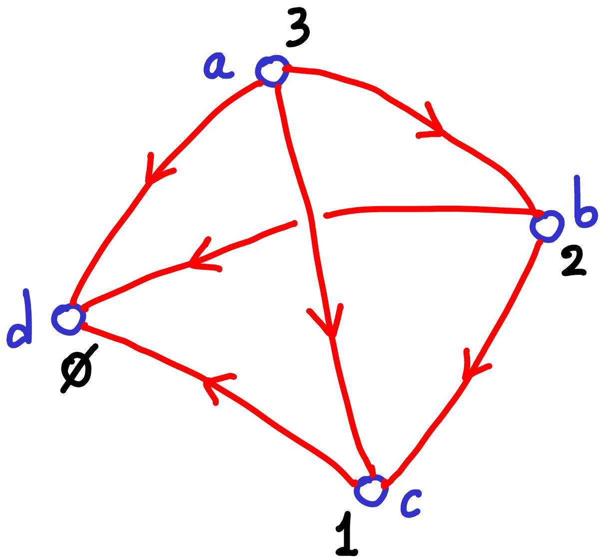

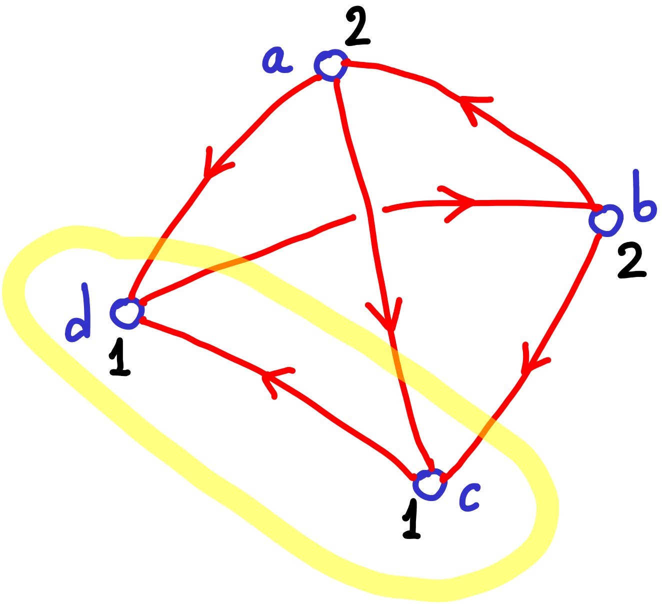

c & d both lost to a

TOURNAMENTS

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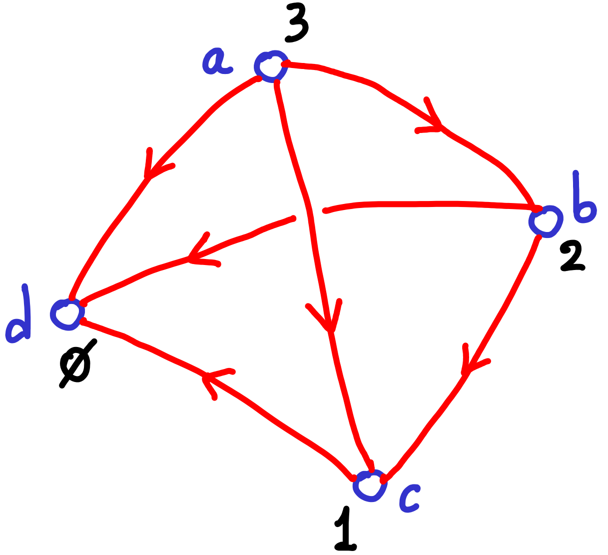

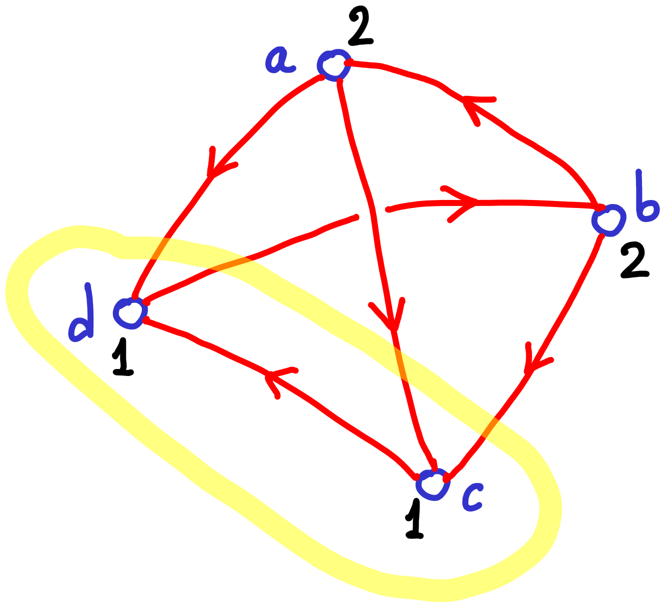

Is it possible for every pair of players to lose to a common opponent?

c & d both lost to a

TOURNAMENTS

n players, everybody plays against everybody

(vertices)

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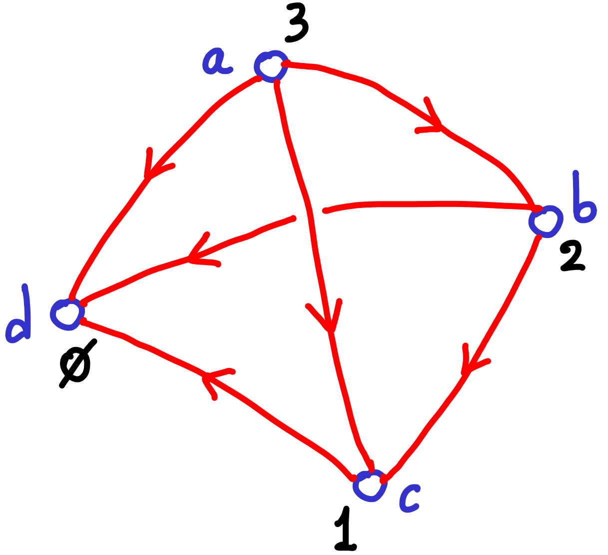
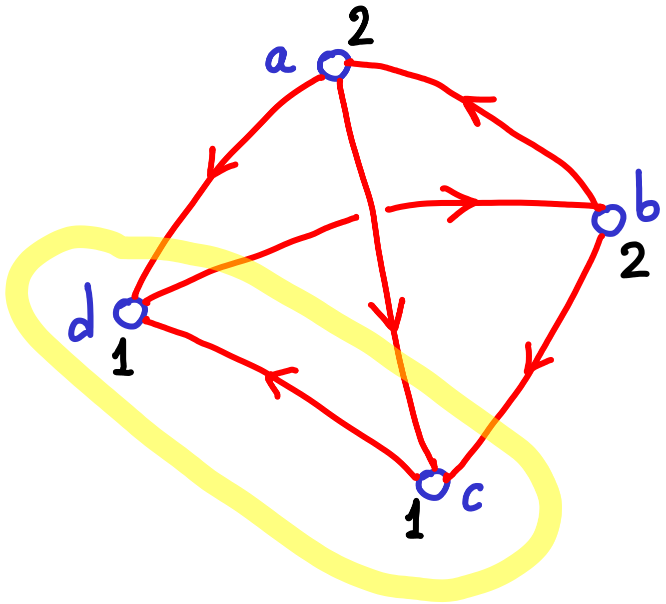




c & d both lost to a

Is it possible for every pair of players to lose to a common opponent?

↓

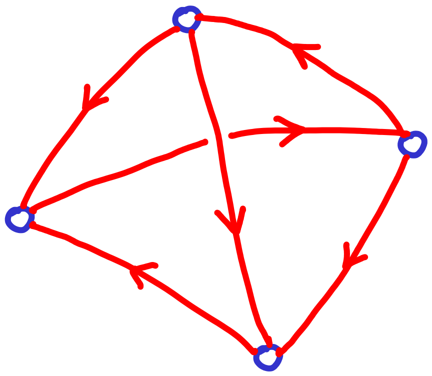
Can we assign directed edges s.t. for every x, y $\exists z$ s.t $\overrightarrow{zx}$ & $\overrightarrow{zy}$?

↓

paradoxical tournament

$n=4$

$\binom{n}{2} = 6$ edges : average # incoming edges per vertex $= 1.5$



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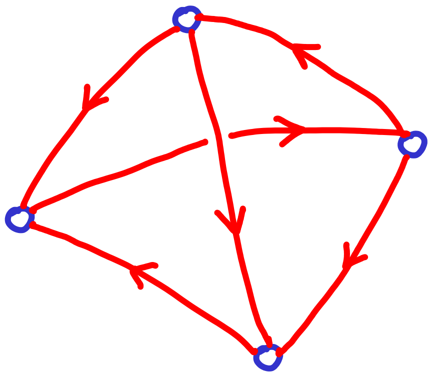

$\hookrightarrow \exists$ vertex with ≤ 1 incoming edge

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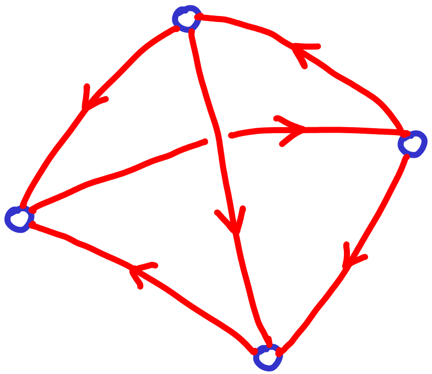


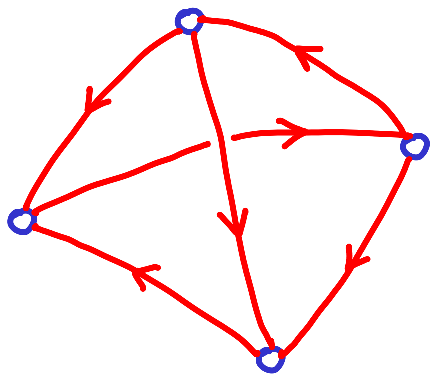
$\hookrightarrow \exists$ vertex with ≤ 1 incoming edge

If only A beats B, then A must beat B,C & B,D

So A beats everyone : no paradoxical tournament

$n=4$

$\binom{n}{2} = 6$ edges : average # incoming edges per vertex $= 1.5$

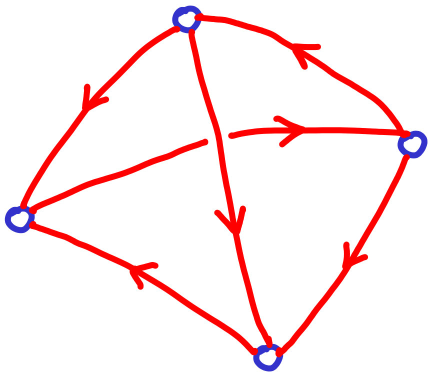


↪ $\exists$ vertex with ≤ 1 incoming edge $\Rightarrow$ impossible

(for any n)

$n=4$

$\binom{n}{2} = 6$ edges : average # incoming edges per vertex $= 1.5$

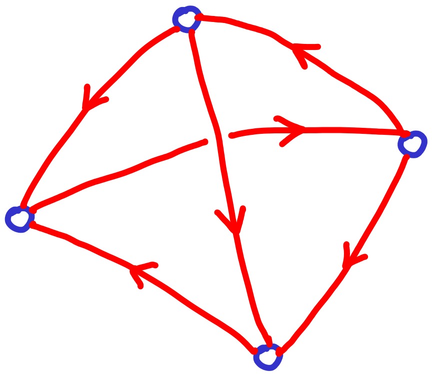


$\hookrightarrow \exists$ vertex with ≤ 1 incoming edge $\Rightarrow$ impossible

$n=5$, 10 edges, average 2 incoming/vertex.

$n=4$

$\binom{n}{2} = 6$ edges : average # incoming edges per vertex $= 1.5$



$\hookrightarrow \exists$ vertex with ≤ 1 incoming edge $\Rightarrow$ impossible

$n=5$, 10 edges, average 2 incoming/vertex.

we know : exactly 2 incoming/vertex
otherwise $\exists$ vertex w/ 1 incoming

$n=4$

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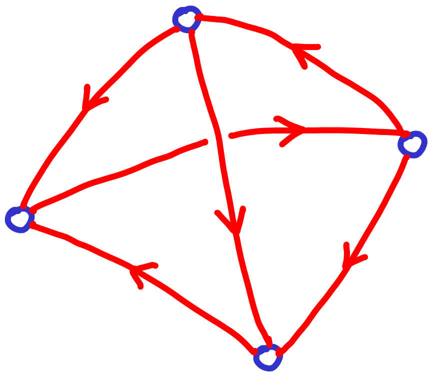

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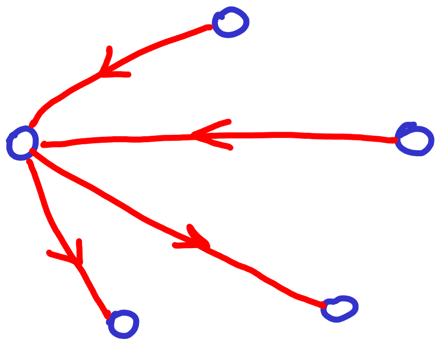

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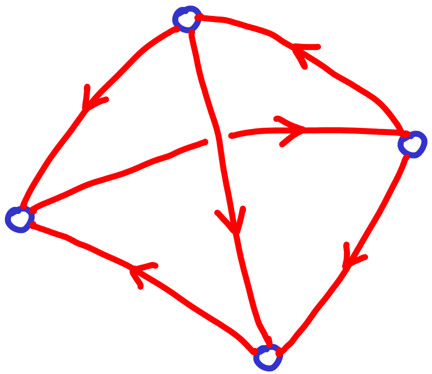

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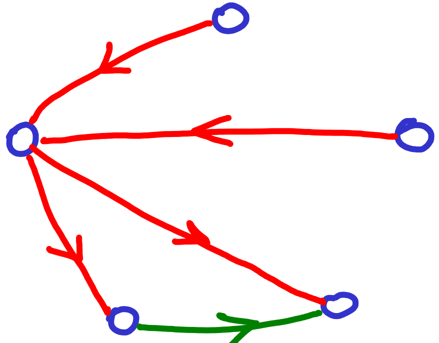
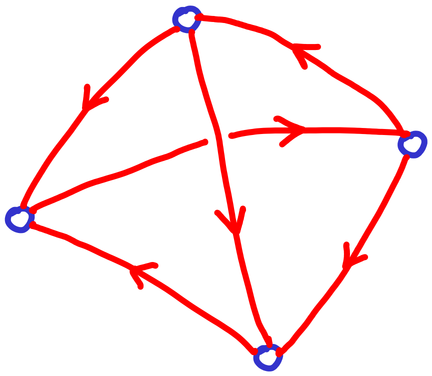


wlog

$n=4$

$\binom{n}{2} = 6$ edges : average # incoming edges per vertex = 1.5



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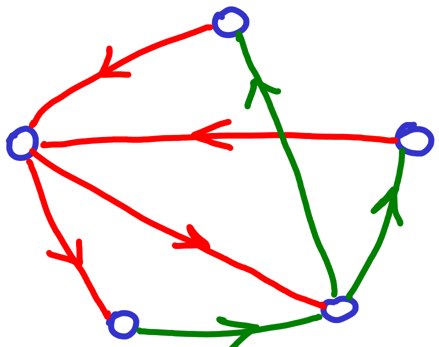

wlog

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$\binom{n}{2} = 6$ edges : average # incoming edges per vertex = 1.5

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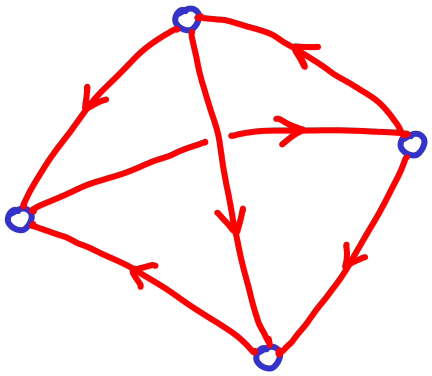

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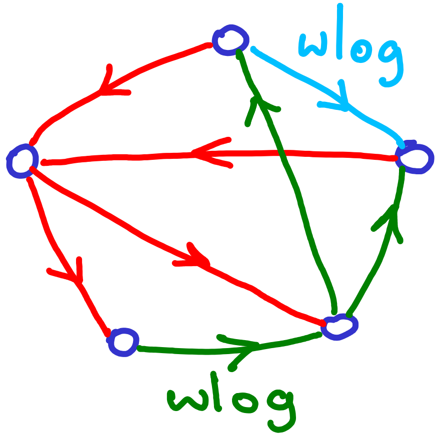

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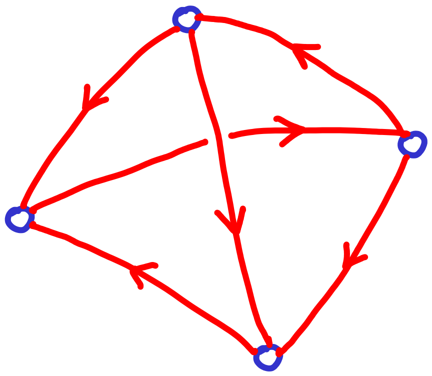

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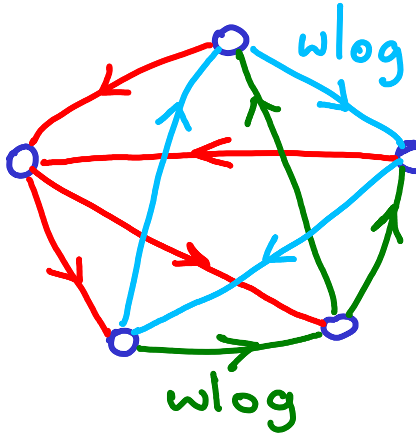


zero pairs

$$\left.\begin{array}{l} n=4 \\ n=5 \end{array}\right\} \text{impossible}$$

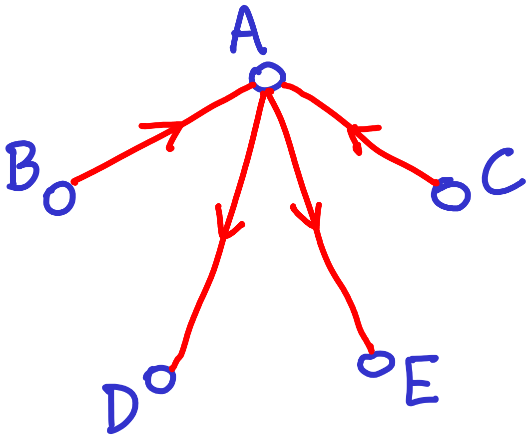
Alternate proof for $n=5$

Can we ever have a vertex with only 2 incoming?

$n=4$
$n=5$ } impossible

Alternate proof for $n=5$

Can we ever have a vertex with only 2 incoming?

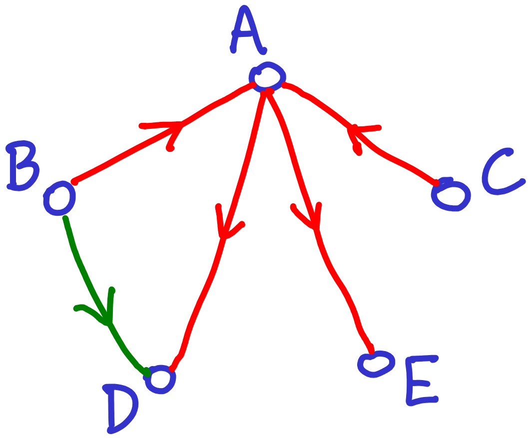


- If A loses only to B & C then ?

$n=4$
$n=5$ } impossible

Alternate proof for $n=5$

Can we ever have a vertex with only 2 incoming?

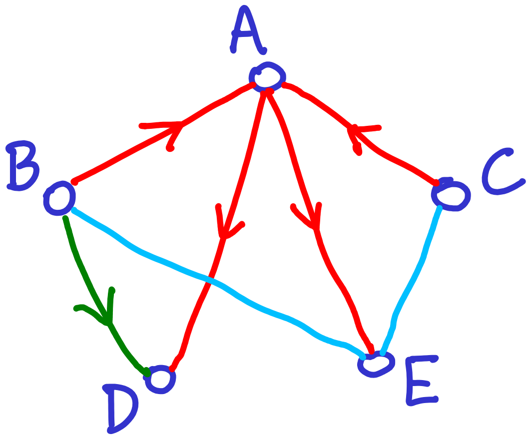


- If A loses only to B & C
- then B or C must beat A,D

$n=4$
$n=5$ } impossible

Alternate proof for $n=5$

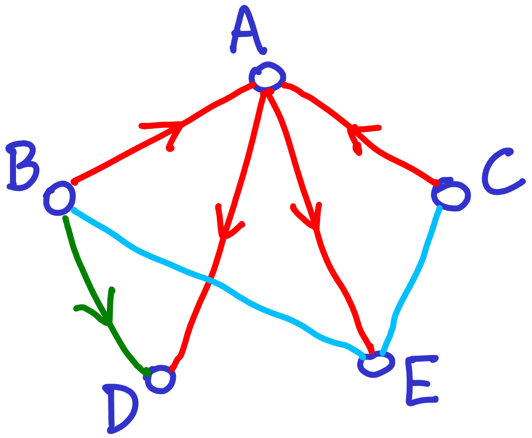
Can we ever have a vertex with only 2 incoming?



- If A loses only to B & C
- then B or C must beat A,D & same for A,E

n=4
n=5 } impossible

Alternate proof for n=5



Can we ever have a vertex with only 2 incoming?

- If A loses only to B & C
- then B or C must beat A,D & same for A,E

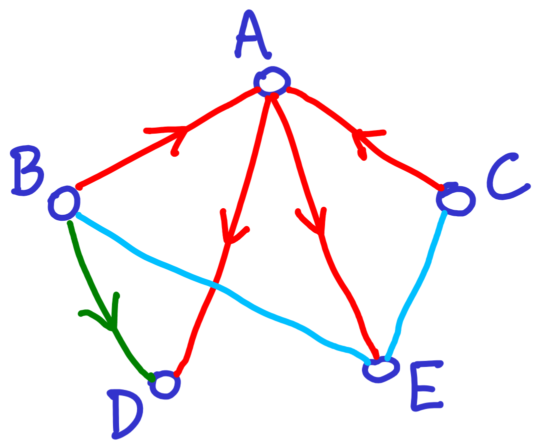
Thus there is no vertex to beat B&C

NO

$n=4$
$n=5$ } impossible

Alternate proof for $n=5$

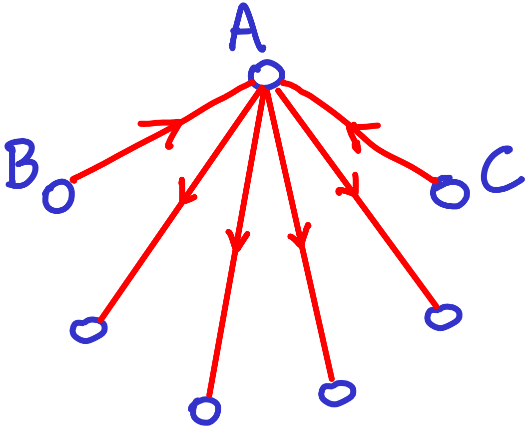
Can we ever have a vertex with only 2 incoming?



- If A loses only to B & C
- then B or C must beat A,D & same for A,E

Thus there is no vertex to beat B & C

NO



Holds for any n: Every vertex must lose to B or C

Thus there is no vertex to beat B & C

$n=6$ ... ?

$\binom{6}{2} = 15$ edges, 2.5 incoming per vertex

$n=6$ ... ?

$\binom{6}{2} = 15$ edges, 2.5 incoming per vertex

$\hookrightarrow$ $\exists$ vertex with ≤ 2 incoming $\Rightarrow$ no 2-paradoxical tournament

$n=6$ ... ?

$\binom{6}{2} = 15$ edges, 2.5 incoming per vertex

$\hookrightarrow$ $\exists$ vertex with ≤ 2 incoming $\Rightarrow$ no 2-paradoxical tournament

---

$n=7$ ... ?

$\binom{7}{2} = 21$ edges, 3 incoming per vertex

$n=6$ ... ?

$\binom{6}{2} = 15$ edges , 2.5 incoming per vertex

↳ $\exists$ vertex with ≤ 2 incoming $\Rightarrow$ no 2-paradoxical tournament

---

$n=7$ ... ?

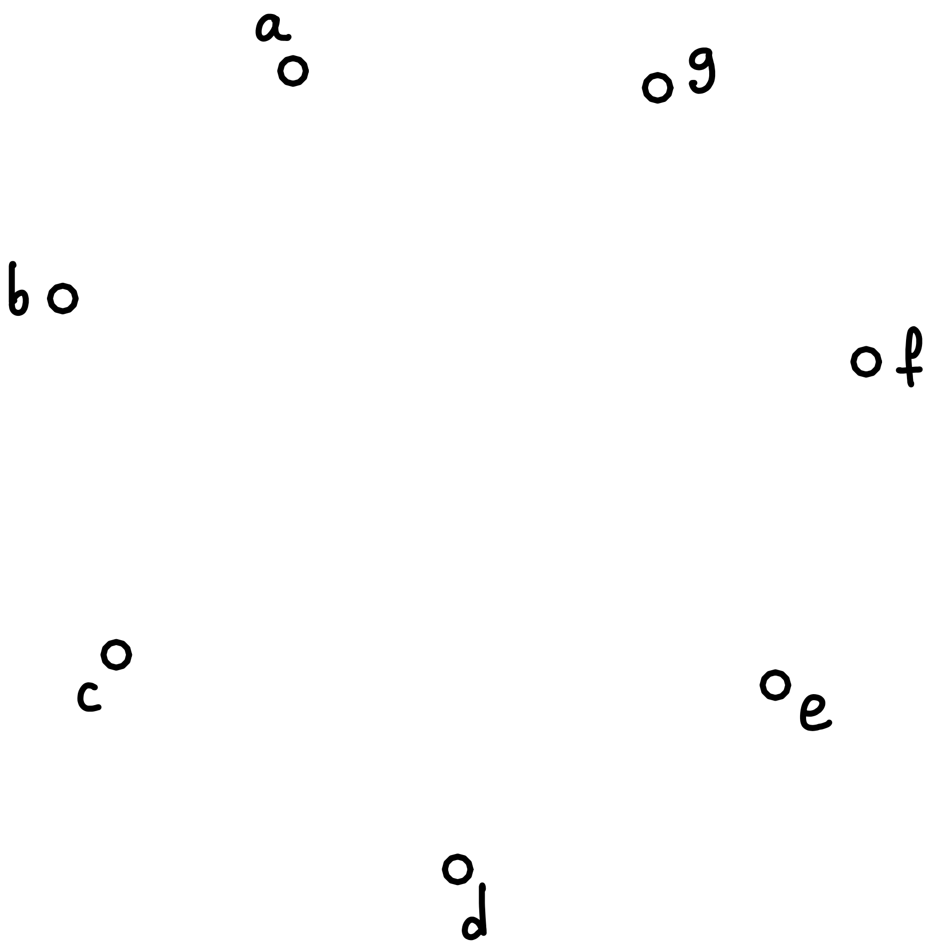
$\binom{7}{2} = 21$ edges , 3 incoming per vertex

Inconclusive

Will it work for 7

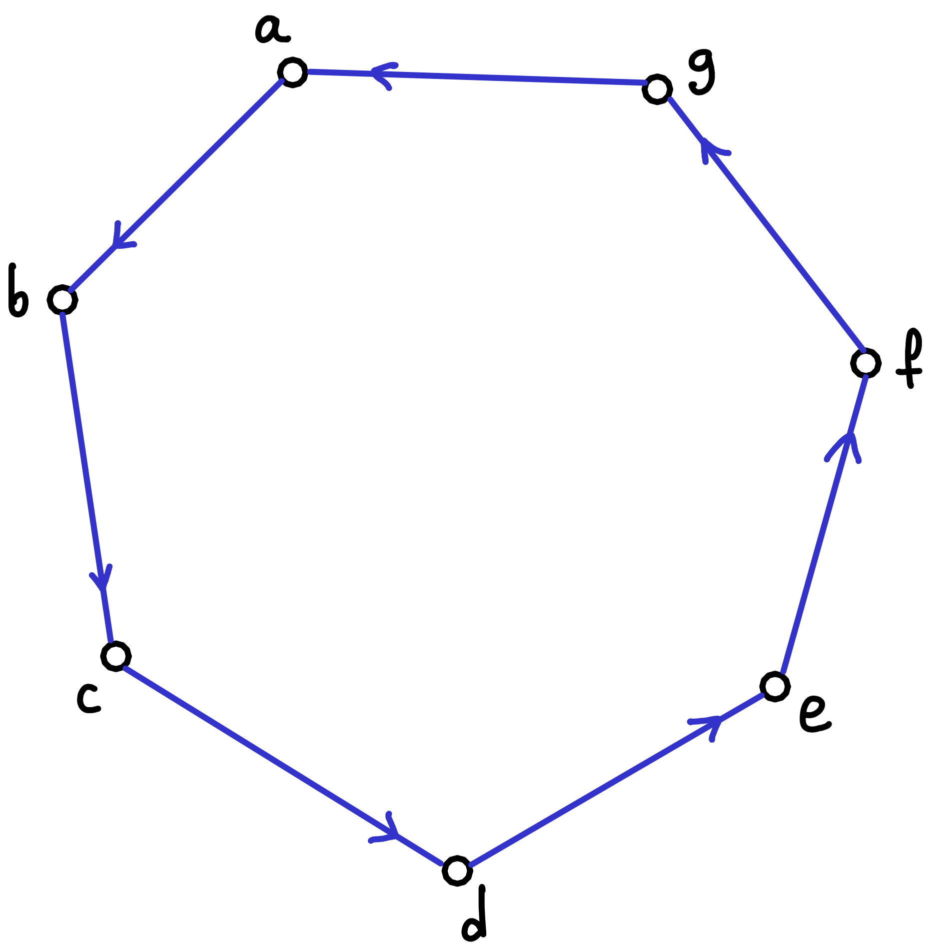
or is there a more refined observation to show it's not possible ?

$n=7$



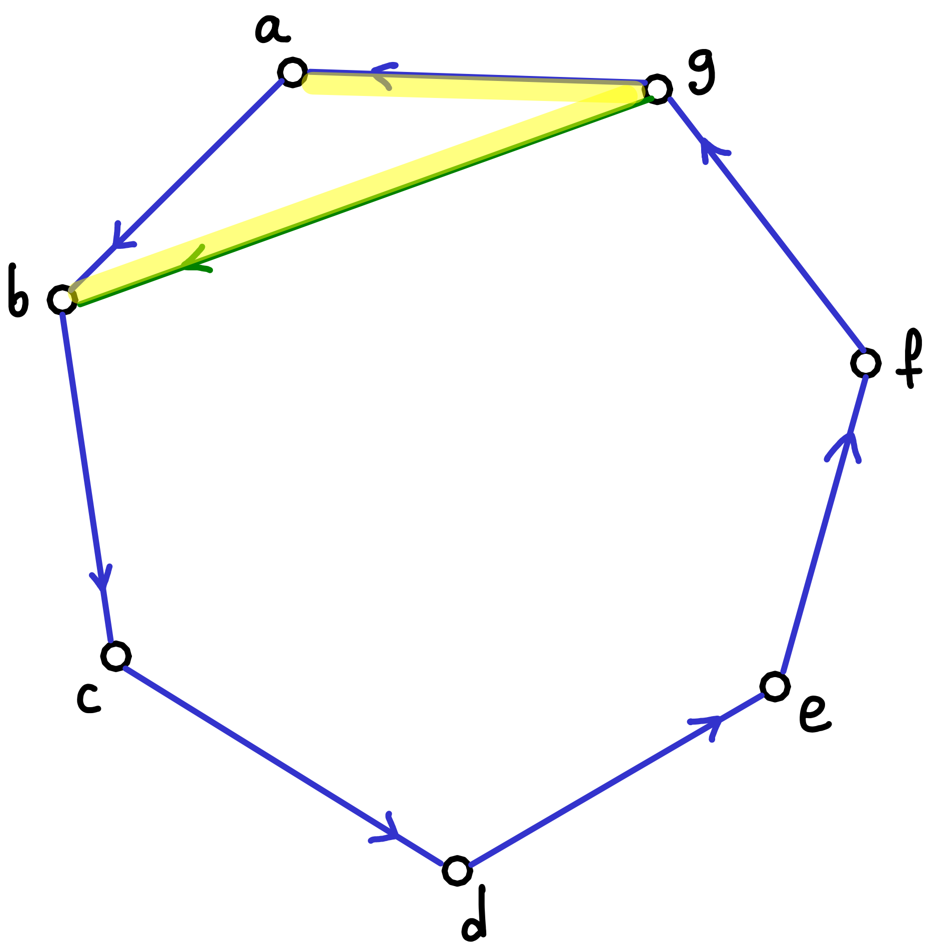
Need exactly 3 incoming per vertex

$n=7$



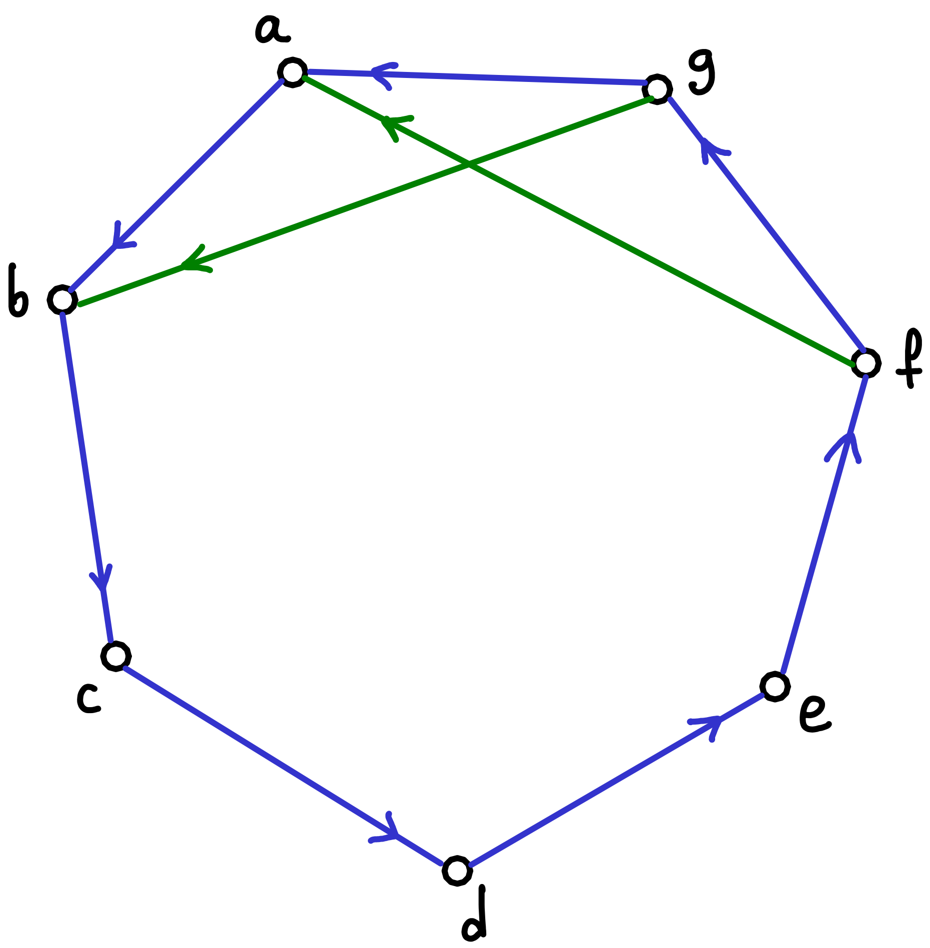
$a, b \leftarrow g$

$n=7$



$a, b \leftarrow g$

$n=7$

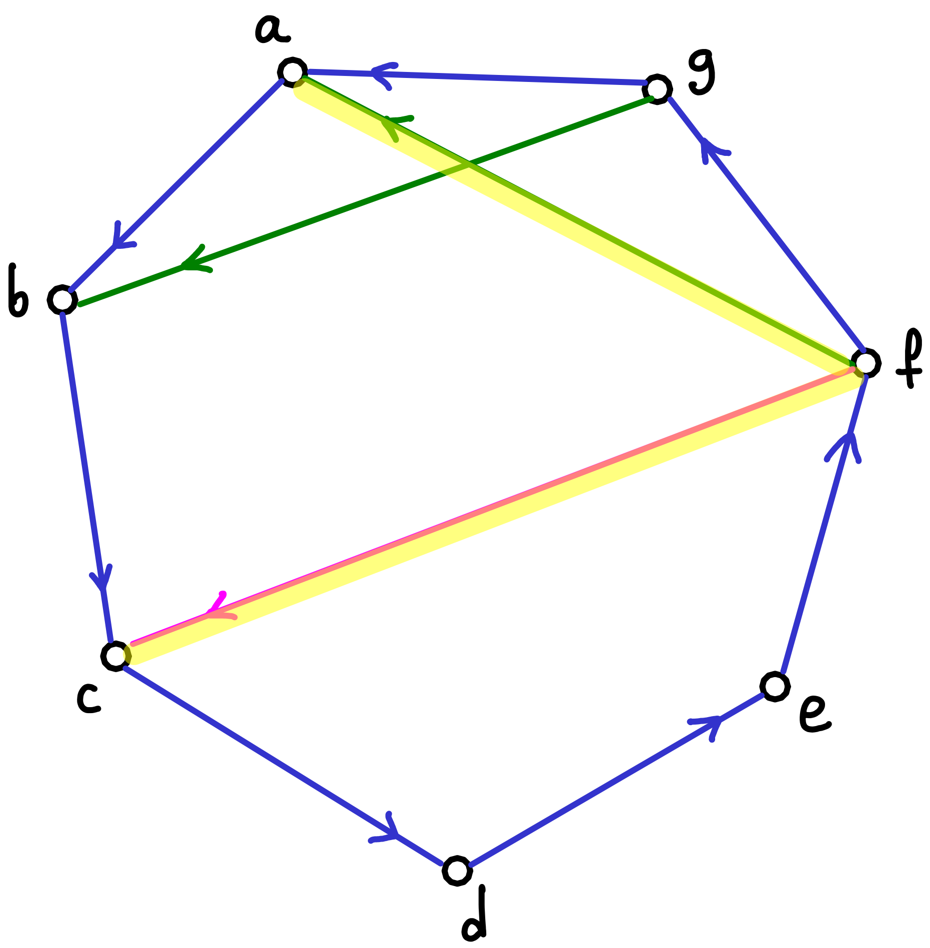


skip 1

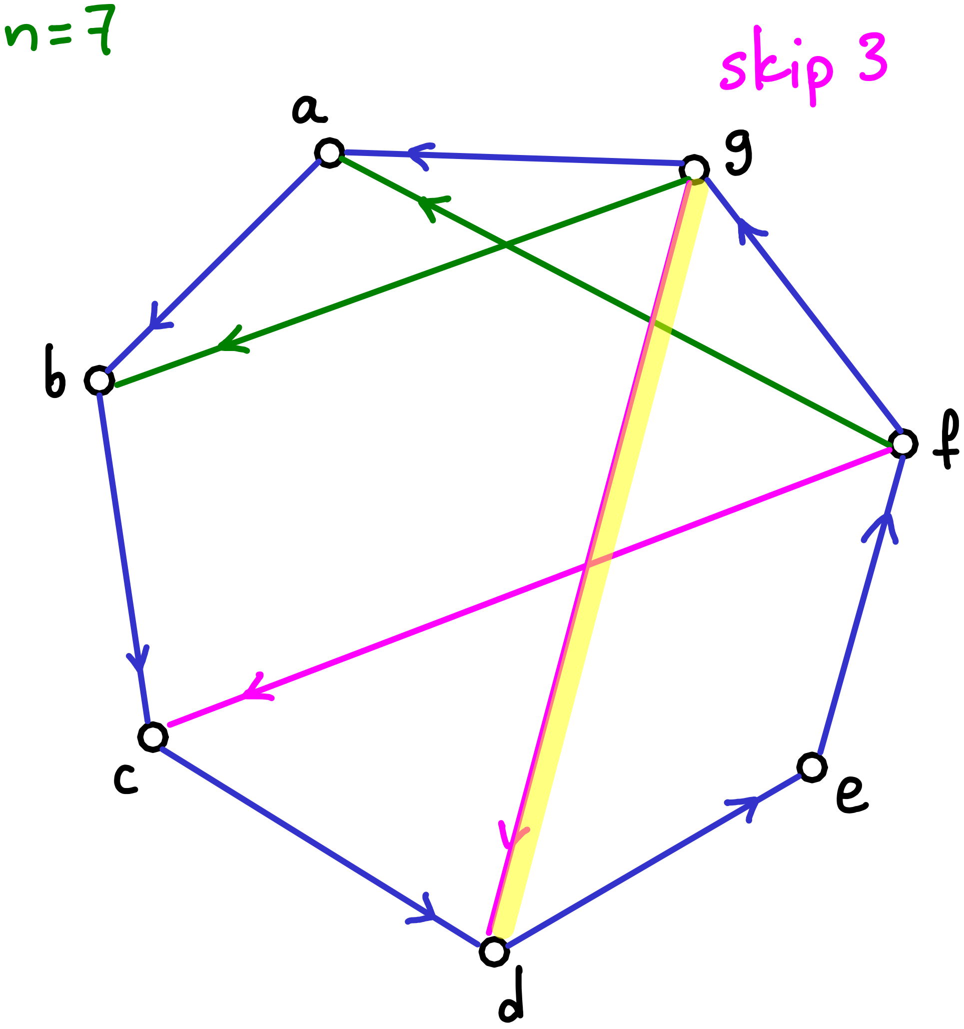
$a, b \leftarrow g$

$a, c \leftarrow f$

$n=7$



skip 3

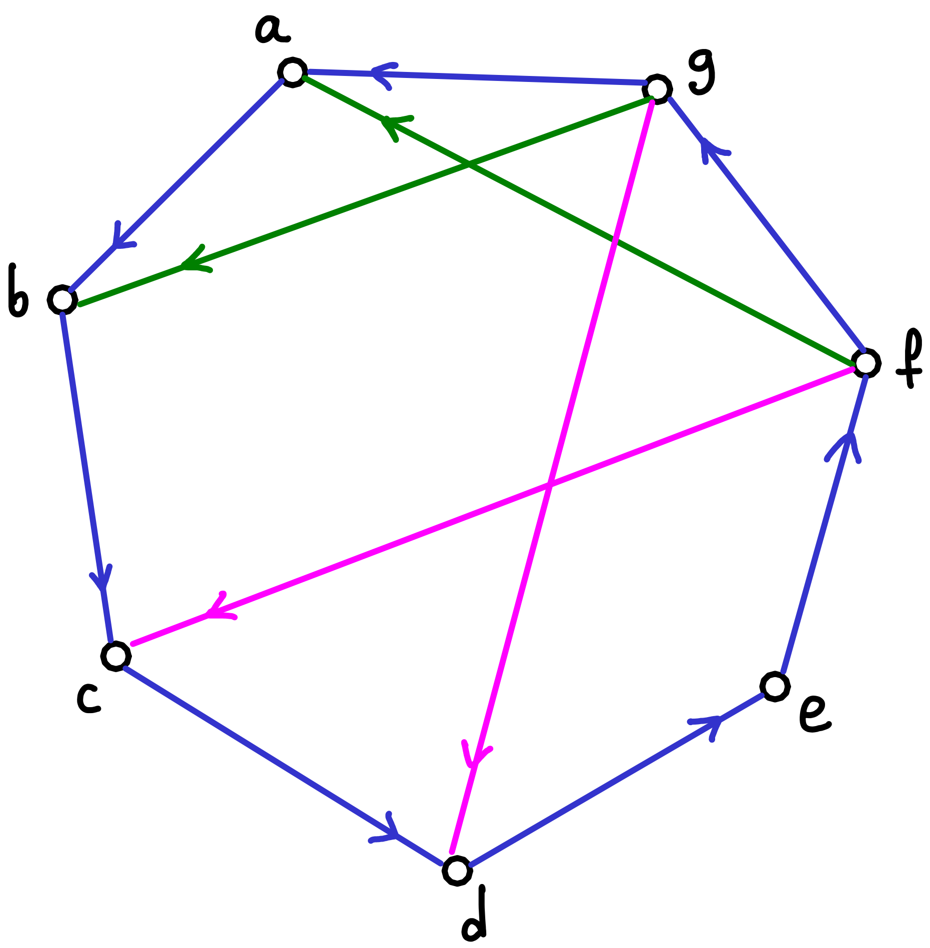


$a, b \leftarrow g$

$a, c \leftarrow f$

$a, d \leftarrow g$

$n=7$



$a,b \longleftarrow g$

$a,c \longleftarrow f$

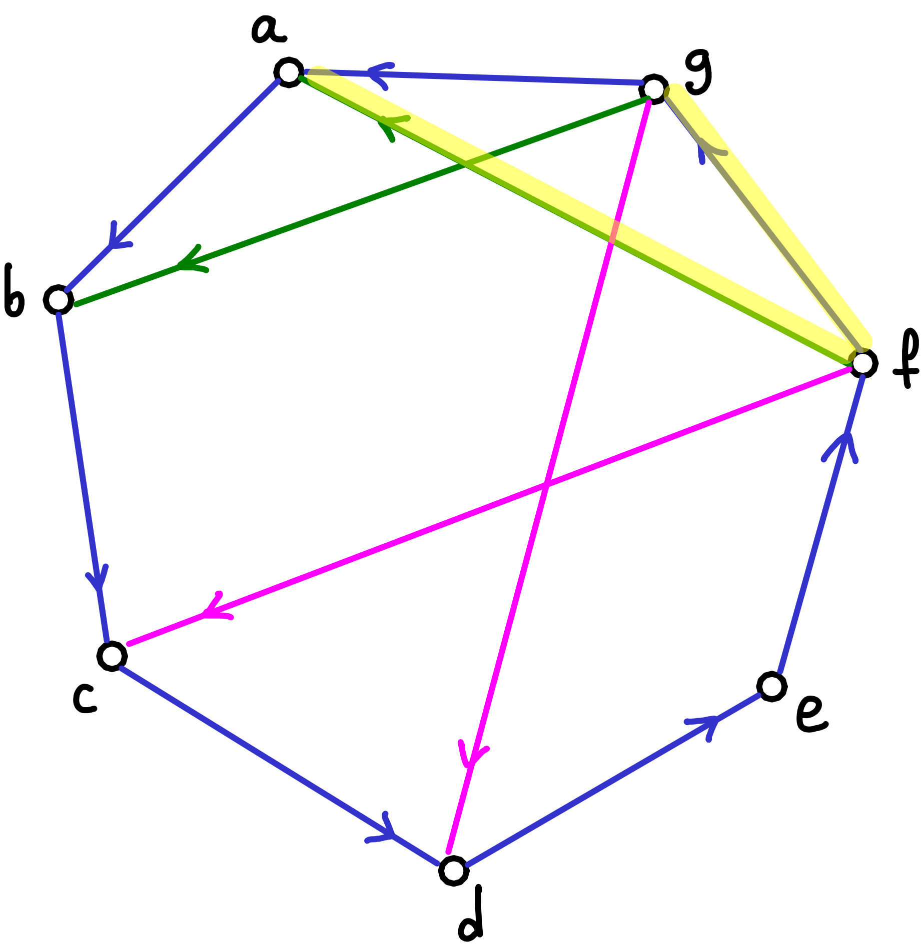
$a,d \longleftarrow g$

a,e ?

a,f ?

a,g ?

$n=7$



$a,b \leftarrow g$

$a,c \leftarrow f$

$a,d \leftarrow g$

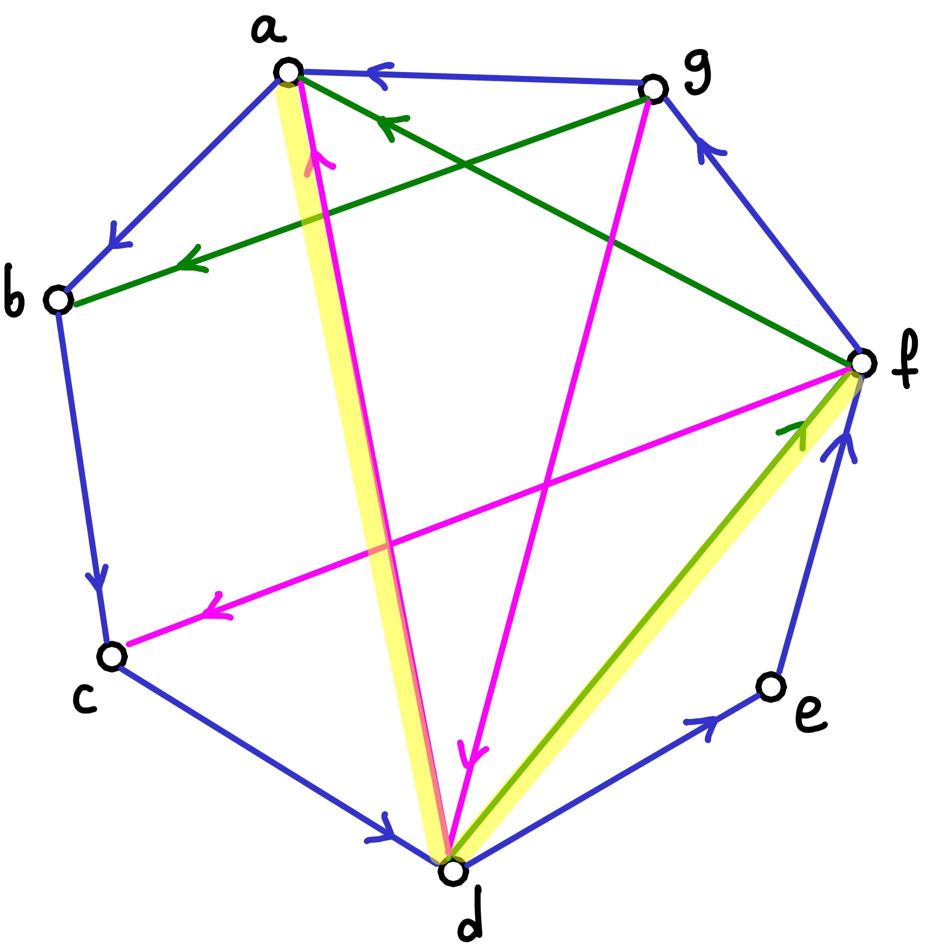
a,e

a,f

$a,g = g,a \leftarrow f$

symmetric

$n=7$



$a,b \longleftarrow g$

$a,c \longleftarrow f$

$a,d \longleftarrow g$

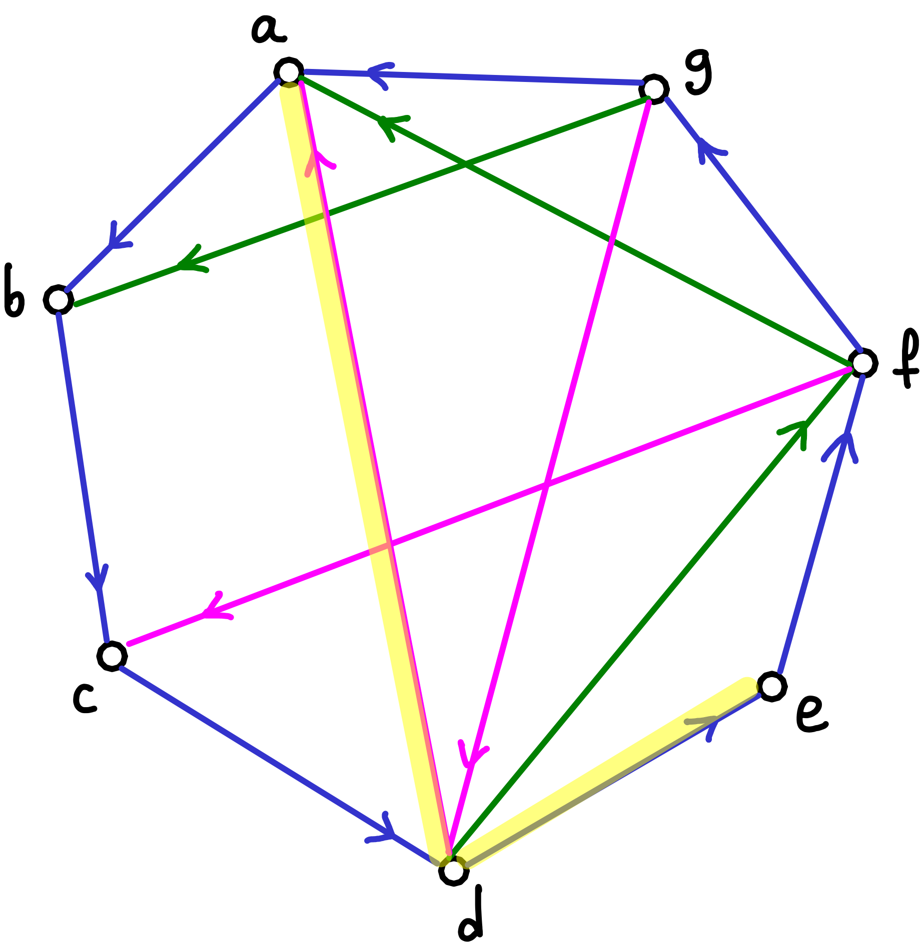
a,e

$a,f = f,a$

$a,g = g,a$

symmetric

$n=7$



$a,b \leftarrow g$

$a,c \leftarrow f$

$a,d \leftarrow g$

$a,e = e,a$

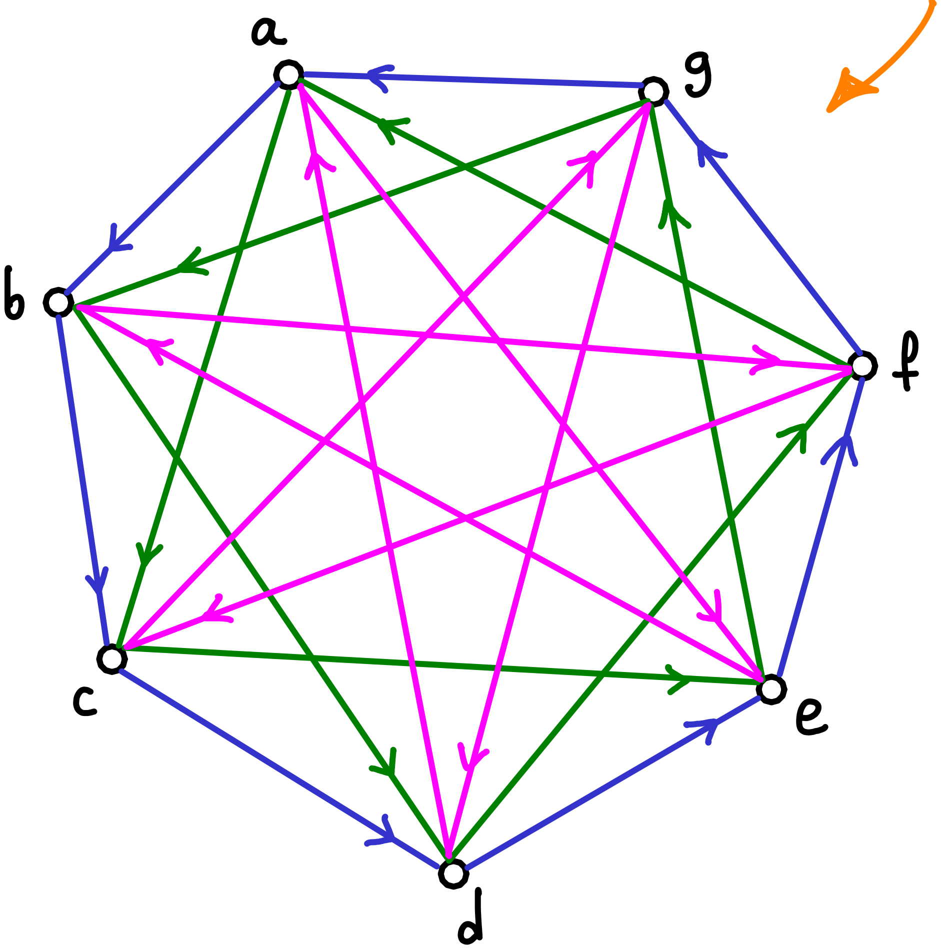
$a,f = f,a$

$a,g = g,a$

symmetric

$n=7$

$\exists$ 2-paradoxical tournament



$a,b \leftarrow g$

$a,c \leftarrow f$

$a,d \leftarrow g$

$a,e = e,a$

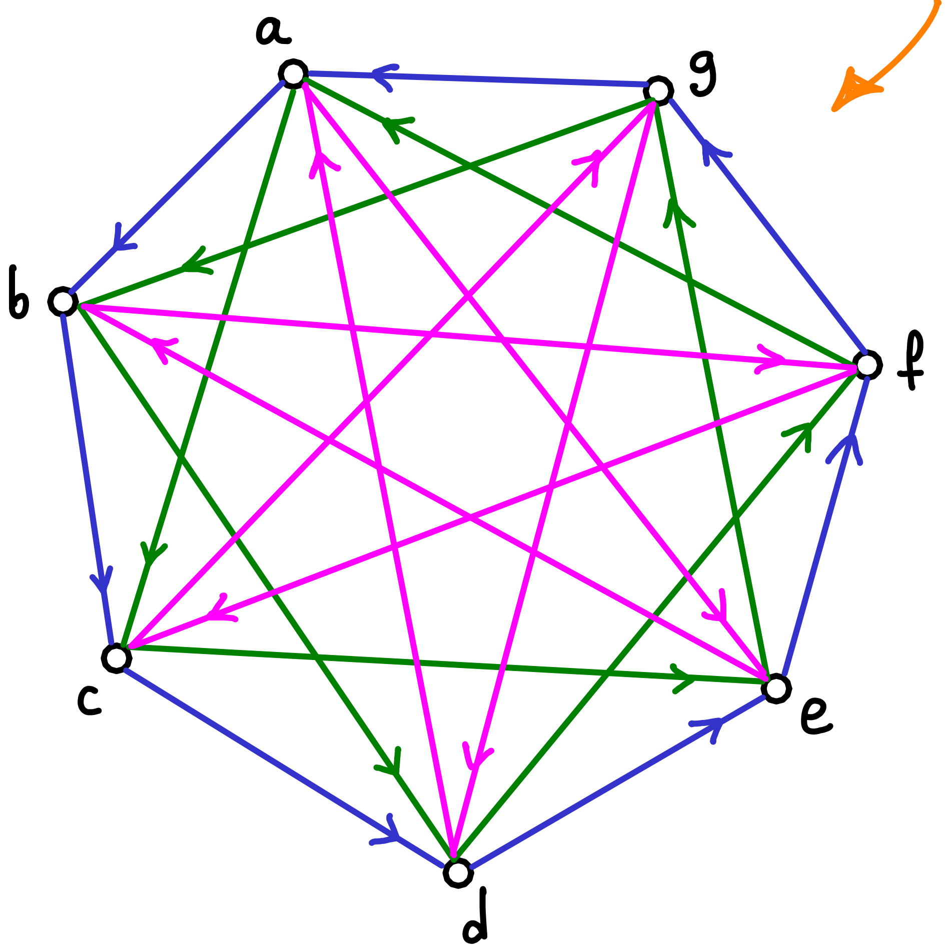
$a,f = f,a$

$a,g = g,a$

symmetric

$n=7$

$\exists$ 2-paradoxical tournament



$a,b \leftarrow g$

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$a,d \leftarrow g$

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$a,f = f,a$

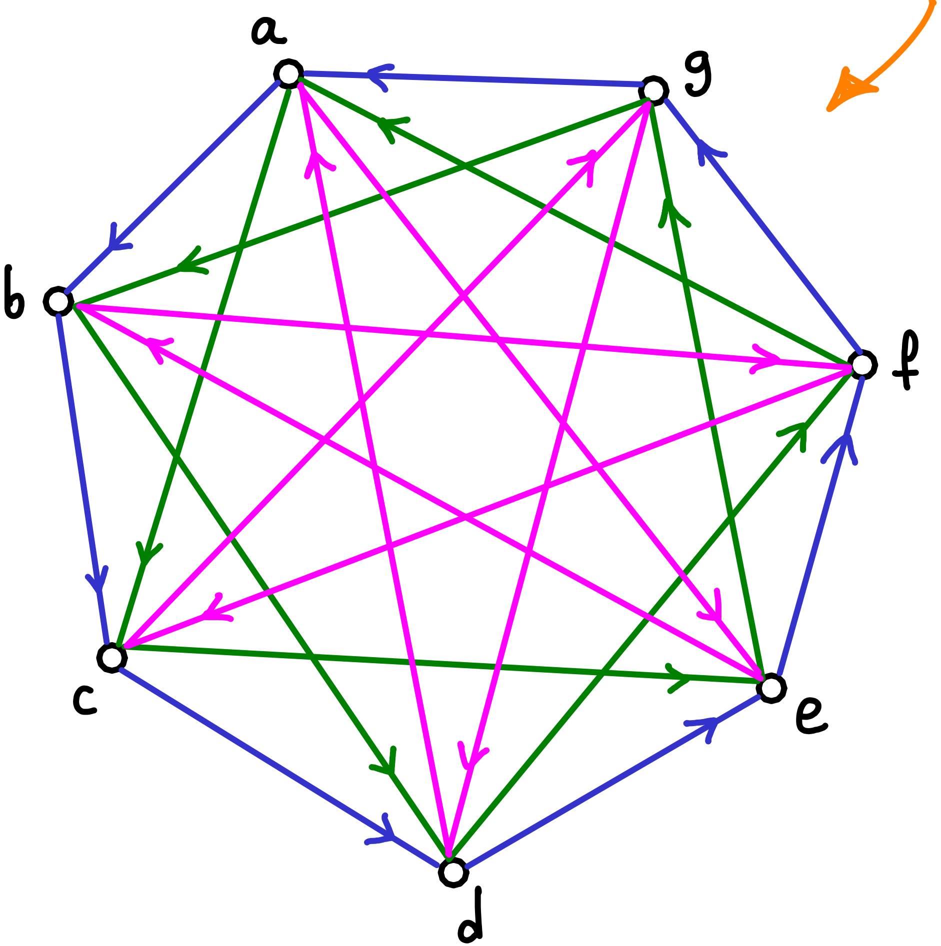
$a,g = g,a$

symmetric

$n>7$?

$n=7$

$\exists$ 2-paradoxical tournament



$a,b \leftarrow g$

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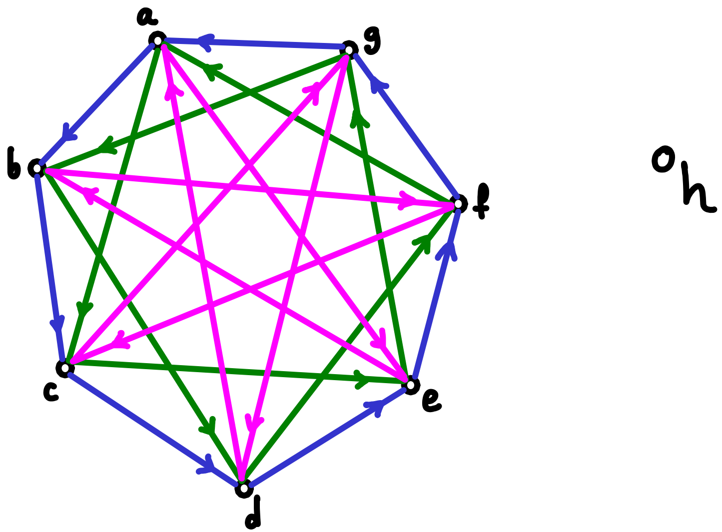

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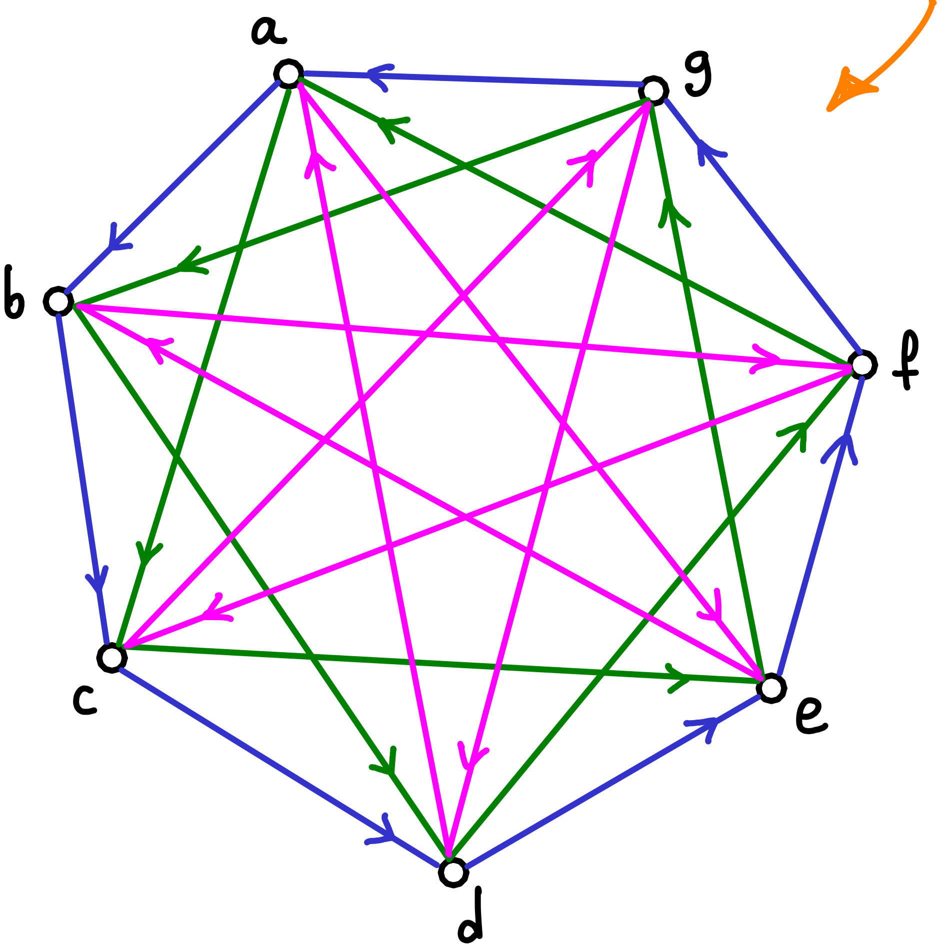

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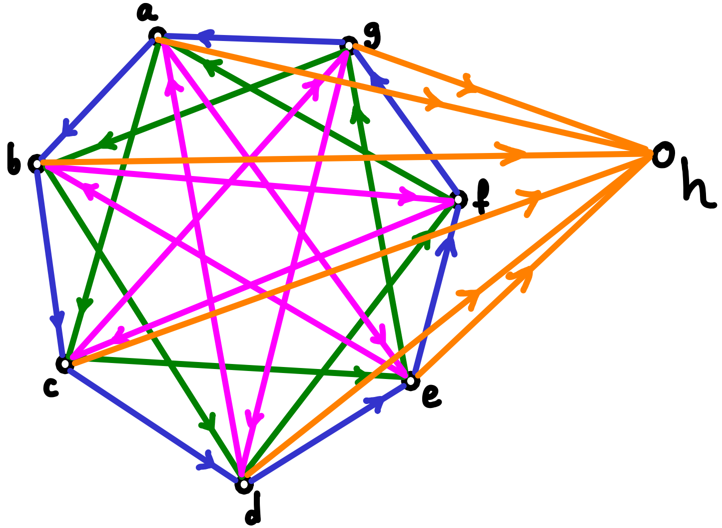

3-paradoxical
TOURNAMENTS
with n players

Is it possible for every <u>triple</u> of players to lose to a common opponent?

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TOURNAMENTS
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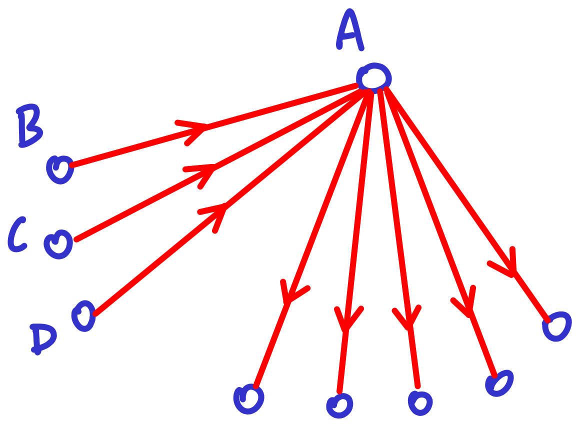

If $\exists$ vertex A with only 3 incoming $(B,C,D \to A)$

then B or C or D must beat all A, X, Y

so B, C, D can't all lose to one vertex

3-paradoxical
TOURNAMENTS
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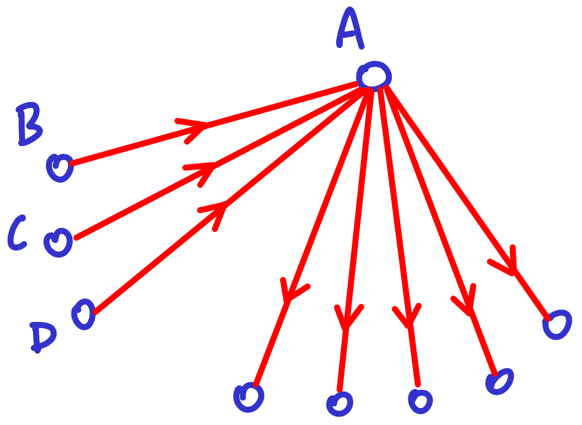

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Need ≥ 9 vertices

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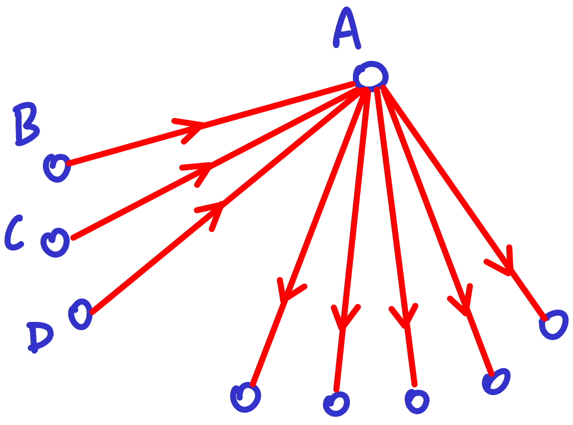

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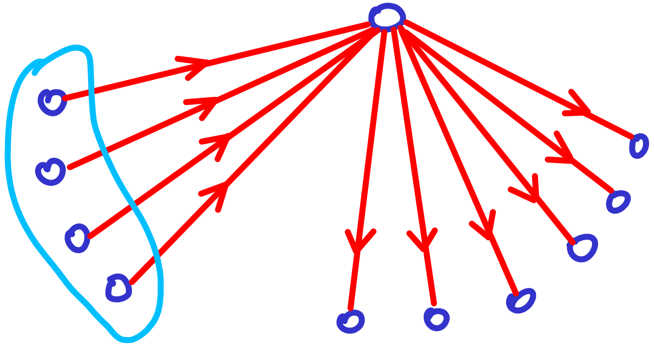

Extend ???

3-paradoxical

TOURNAMENTS

with n players

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Let each edge be directed randomly

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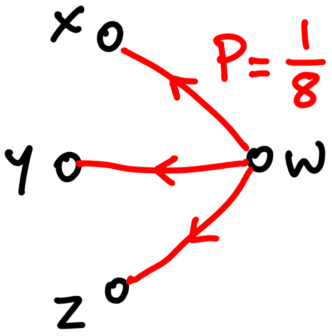

For any x,y,z: $P(x,y,z \text{ all lose to specific opponent } w) = \frac{1}{8}$

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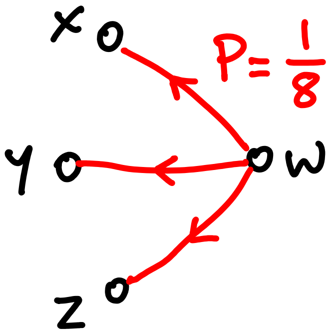

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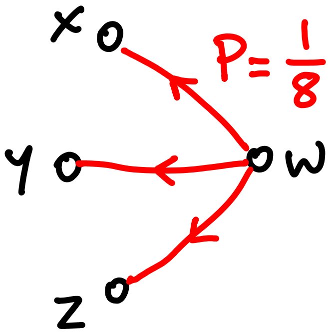

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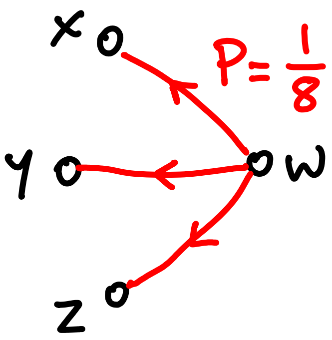

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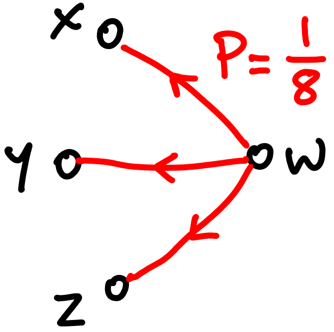

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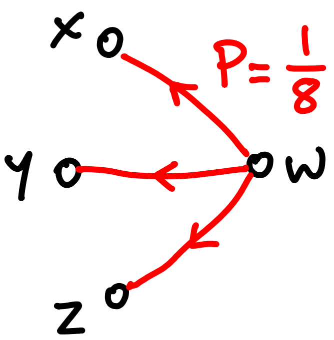

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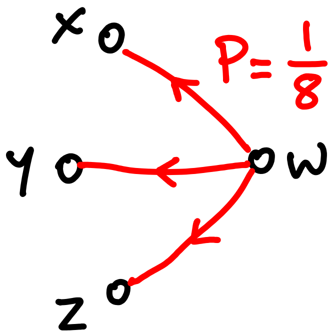

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// $P(A \cup B) \leq P(A) + P(B)$

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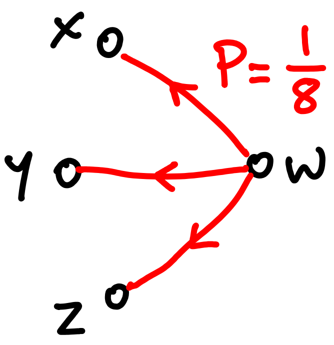

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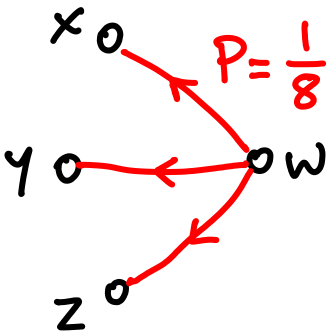

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... so?

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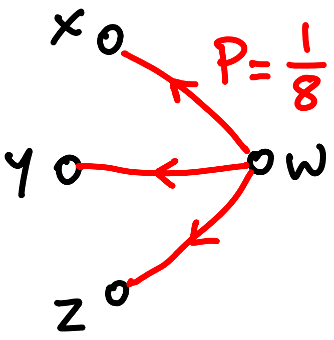

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Yes, if 91 players, $\exists$ 3-paradoxical tournament

k-paradoxical
TOURNAMENTS
with n players

For what k is it possible for every subset of k players to lose to a common opponent?

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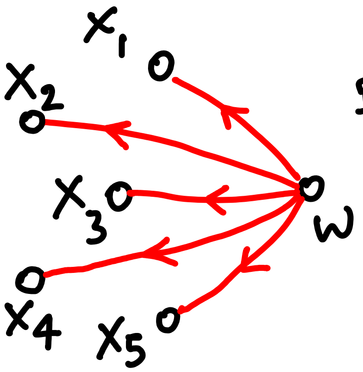

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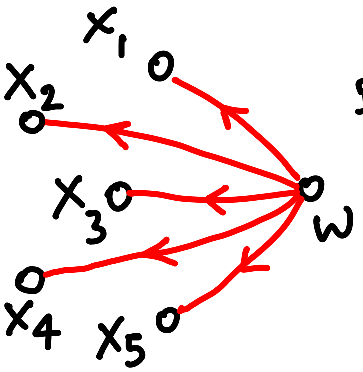

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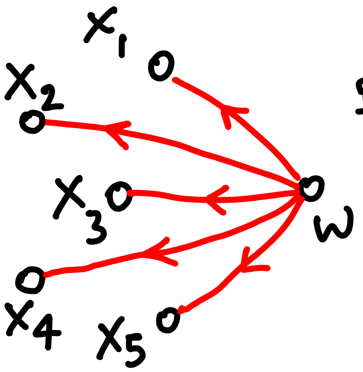

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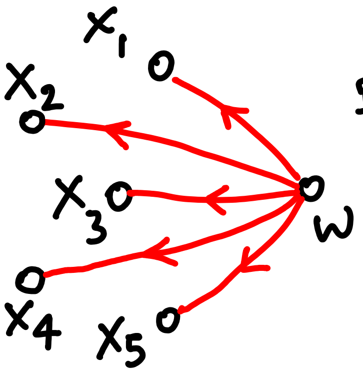

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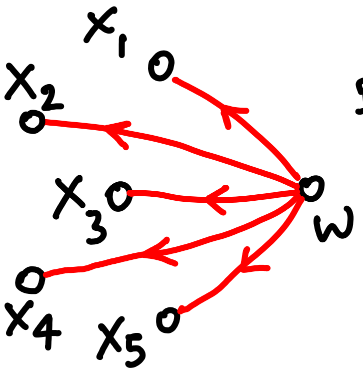

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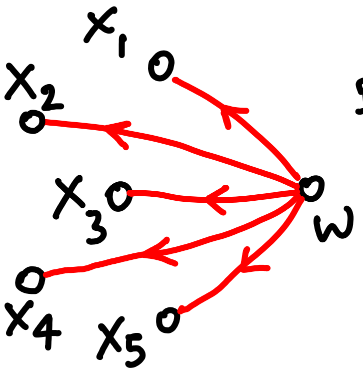

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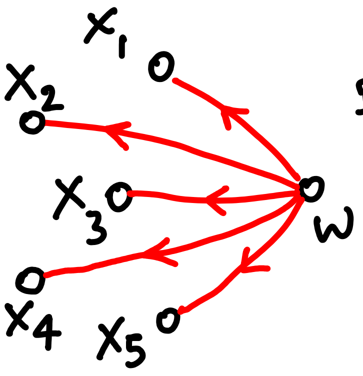

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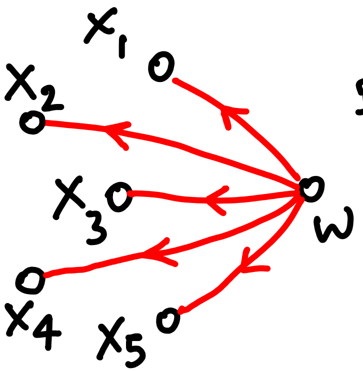

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(For all k, $\exists\, n$)

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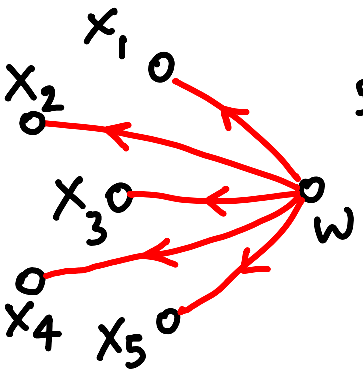

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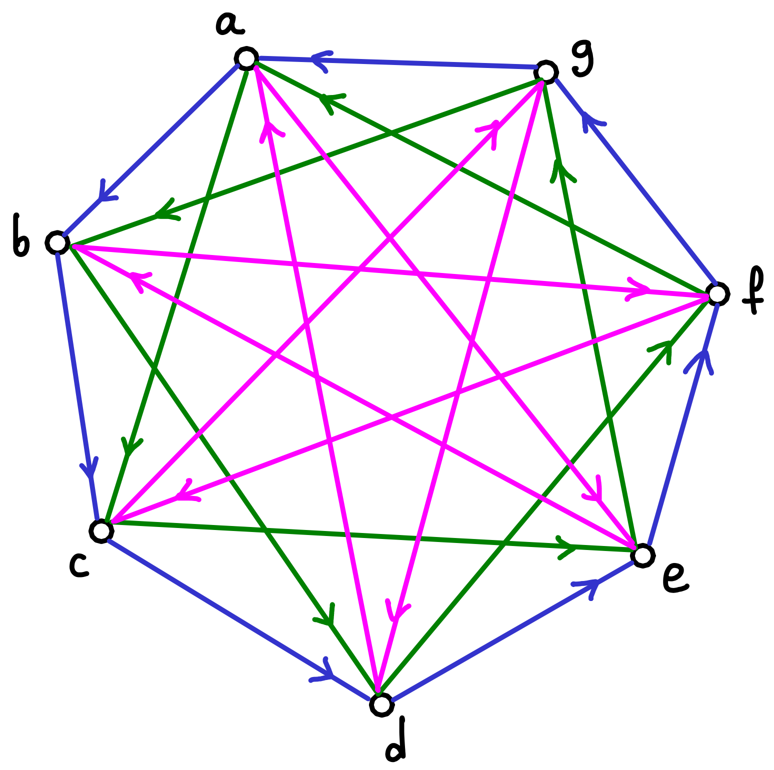
(For all k, $\exists\, n$)

// also known: if $n < c \cdot k \cdot 2^k$, no k-paradoxical

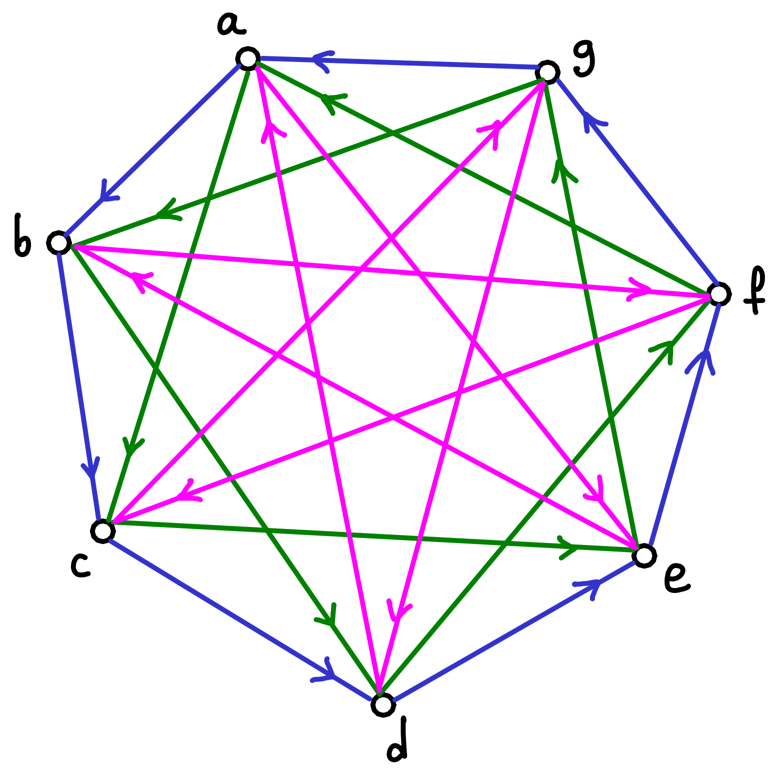
How many Hamiltonian paths could we get in a tournament (graph)?



How many Hamiltonian paths could we get in a tournament (graph)?

Consider any permutation π of vertices

$P(\pi \text{ is a H-path}) = ?$



How many Hamiltonian paths could we get in a tournament (graph)?

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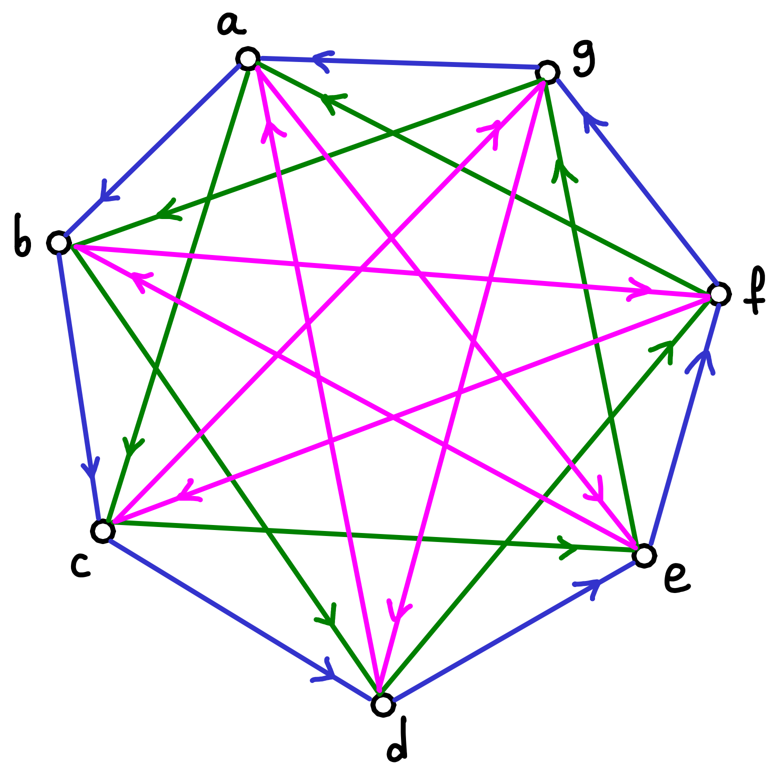

$$P(\pi \text{ is a H-path}) = \left(\frac{1}{2}\right)^{n-1}$$

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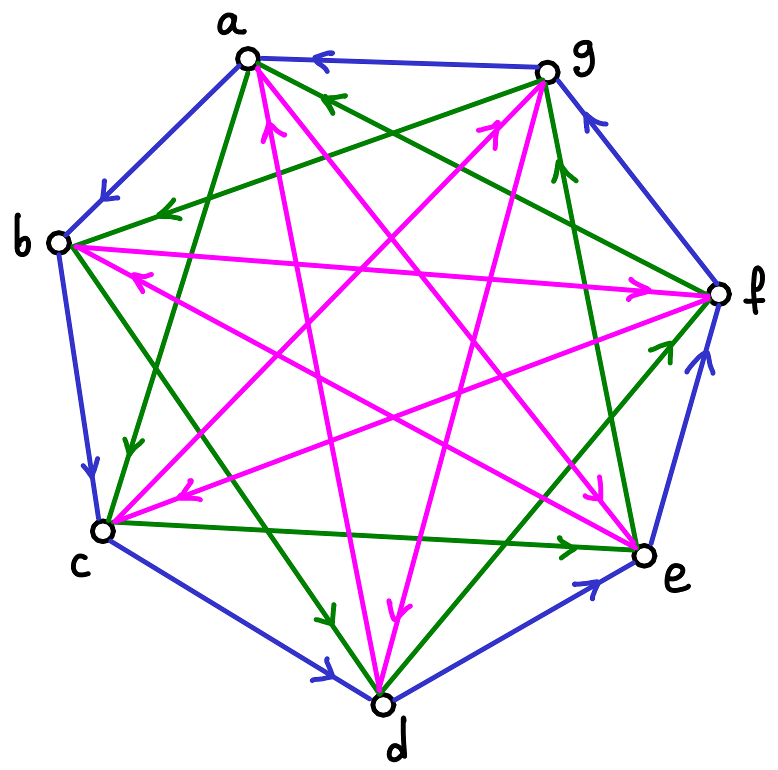

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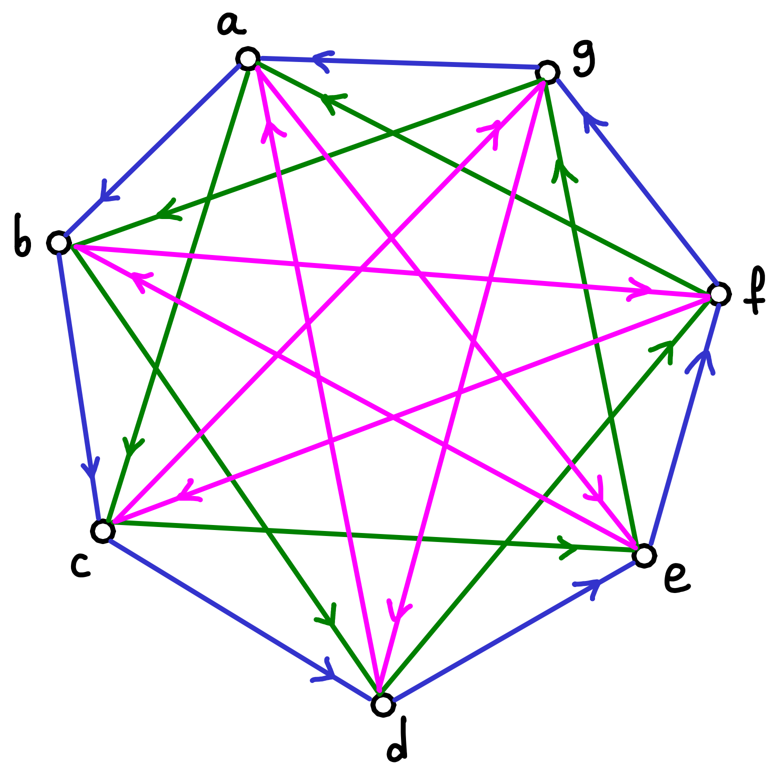

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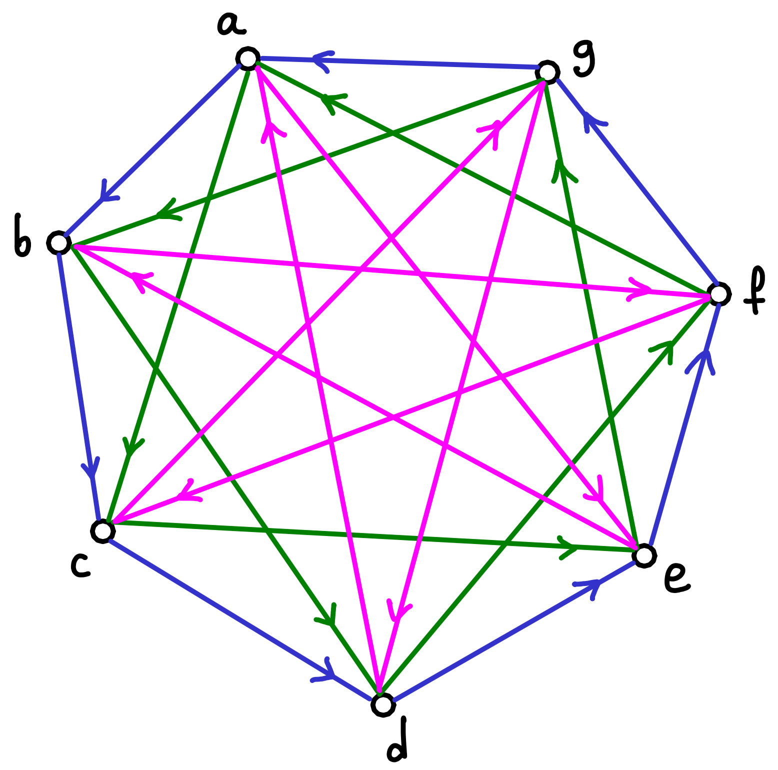

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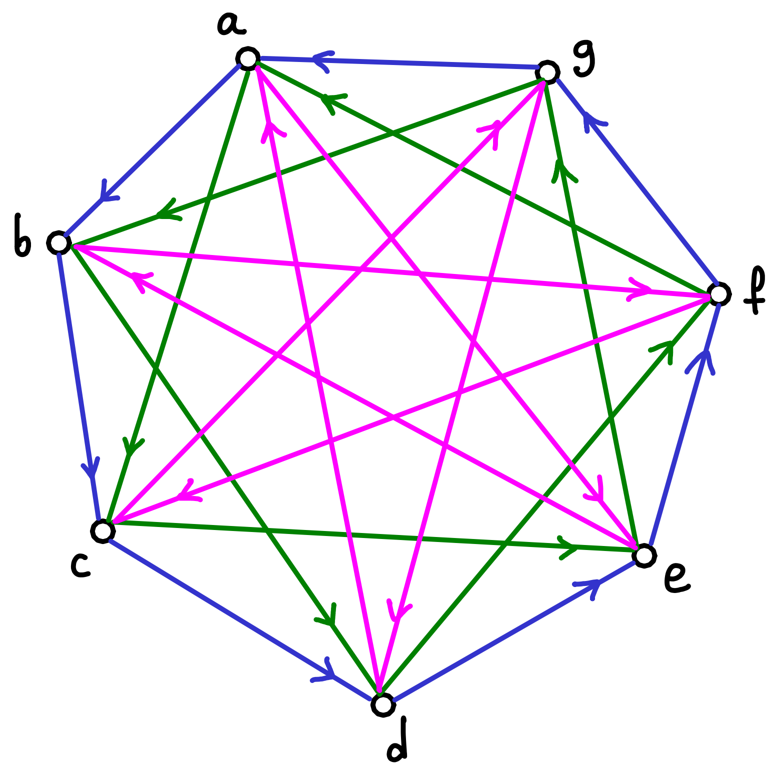

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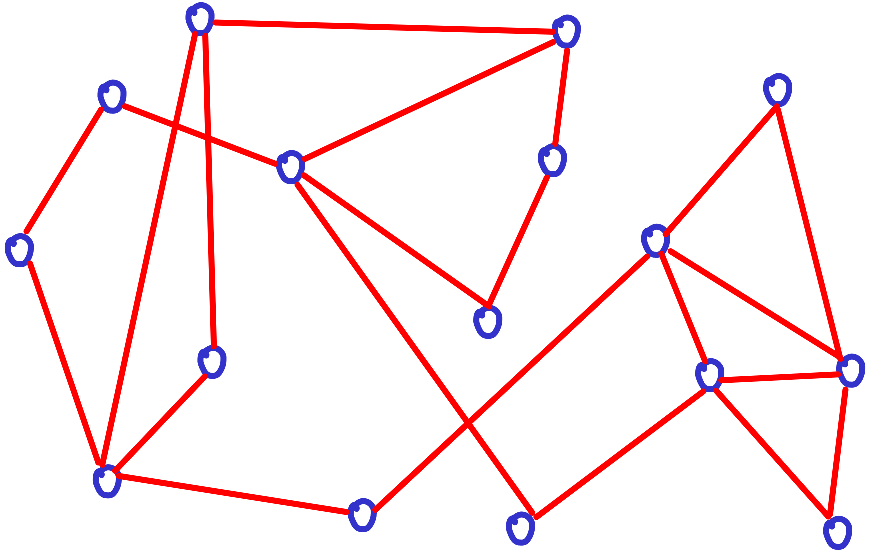
$\exists$ tournament with $\geq \frac{n!}{2^{n-1}}$ Hamiltonian paths

$|V| = 16$

$|E| = 22$



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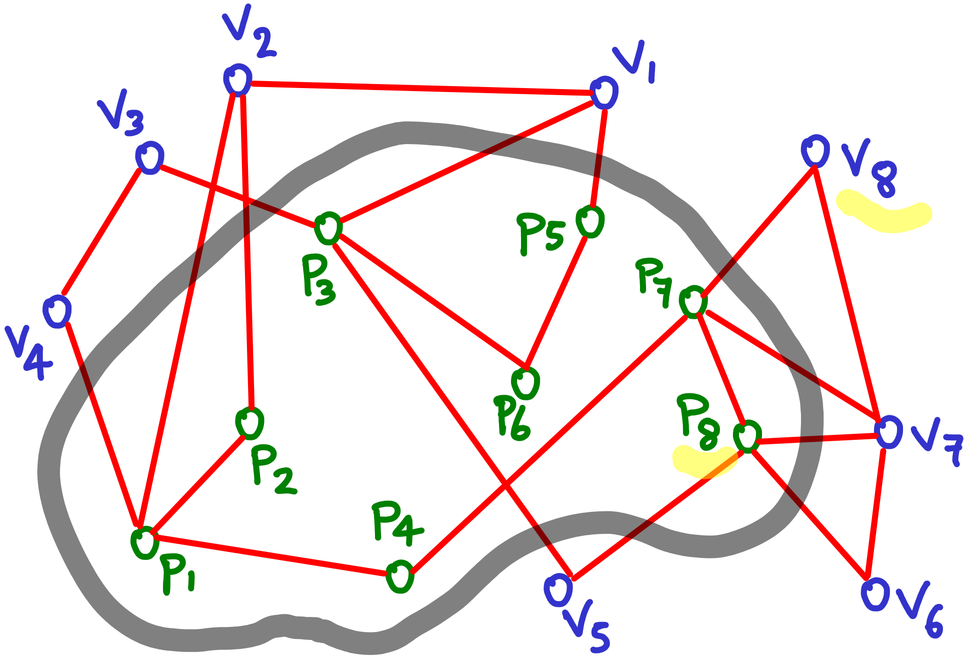

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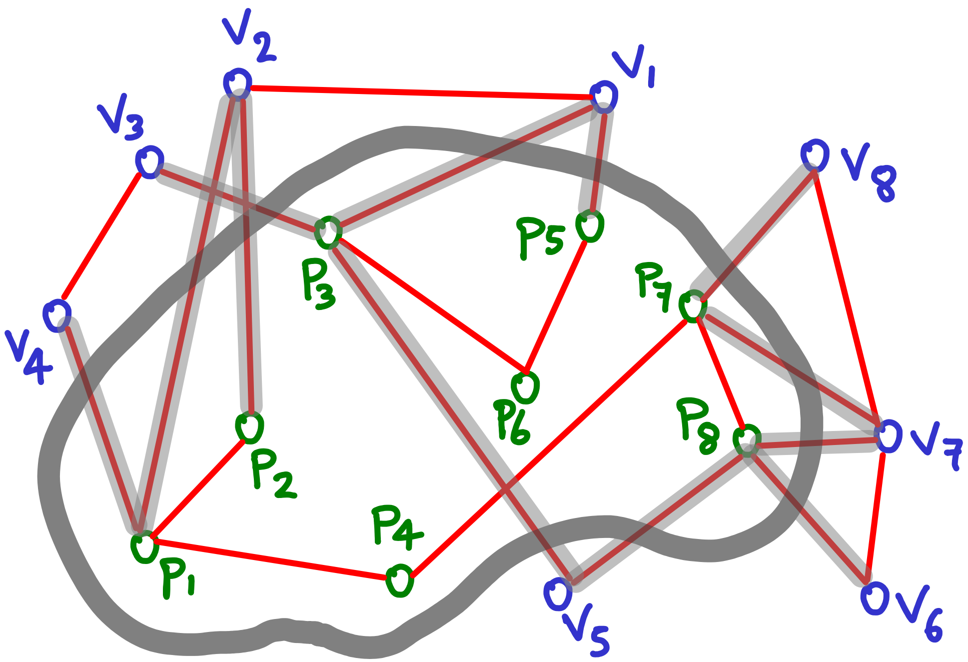
12 edges cross this cut



Claim: every $G = \{V, E\}$ $(V = 2n)$ admits a cut such that the cut partitions V into two subsets of size $\frac{V}{2}$ & crosses $\geq \frac{E}{2}$ edges

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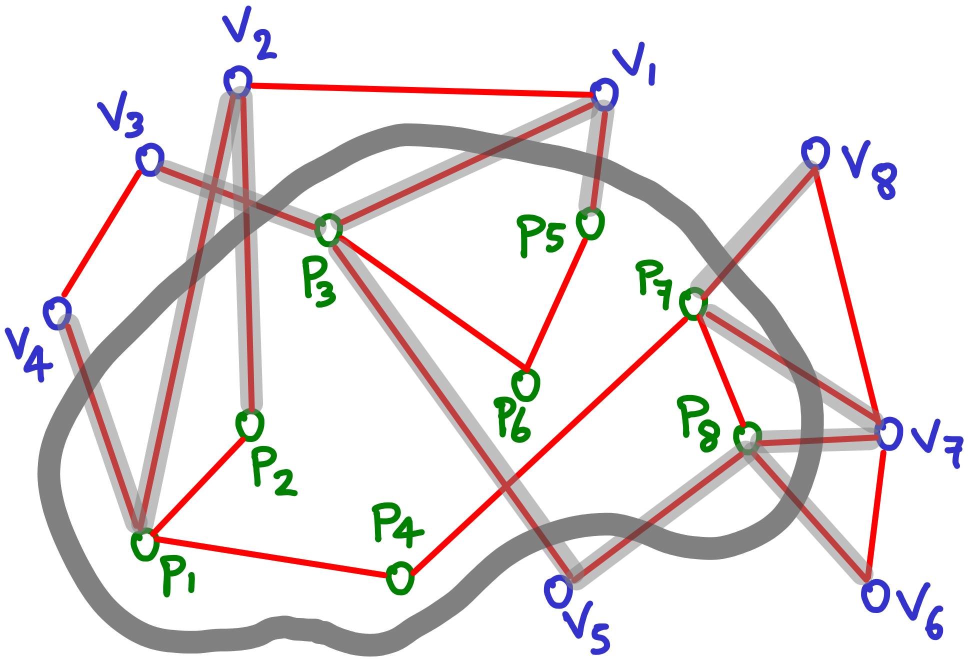

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Every graph contains a large bipartite subgraph

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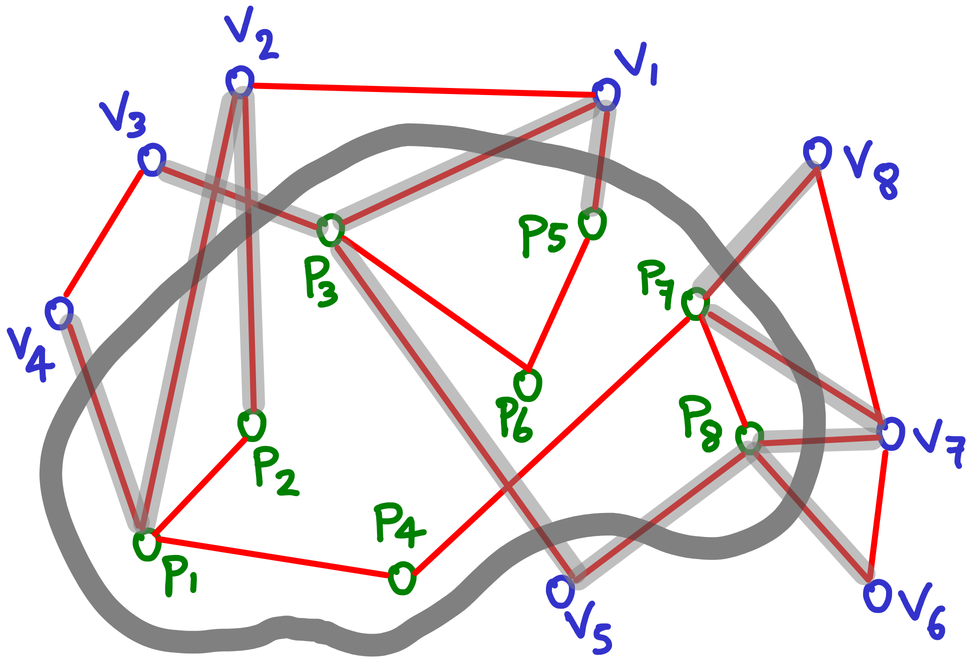
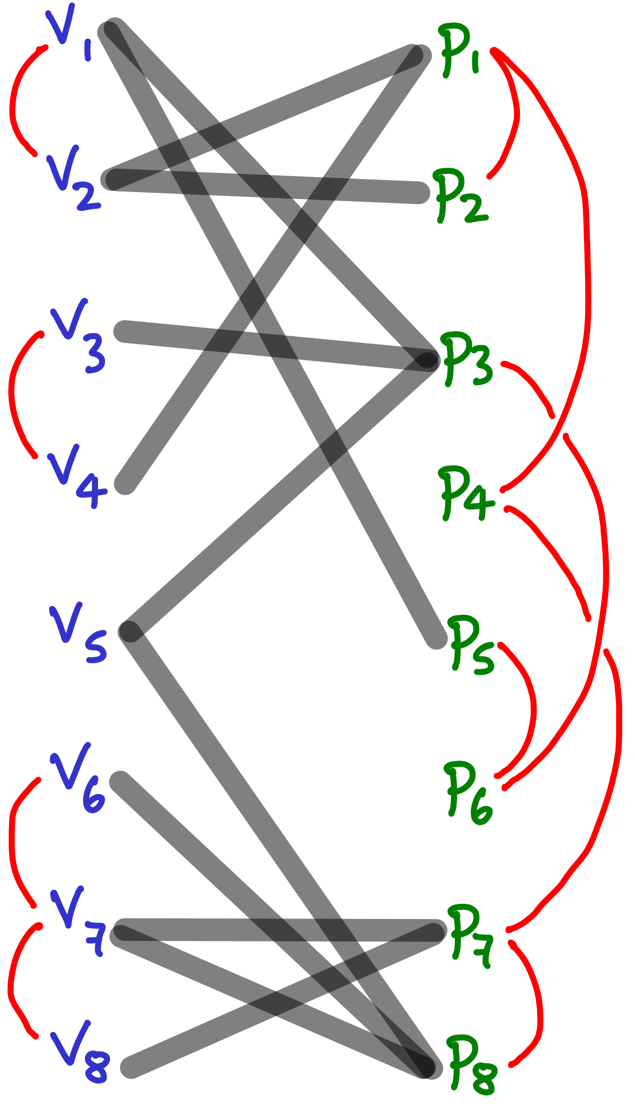
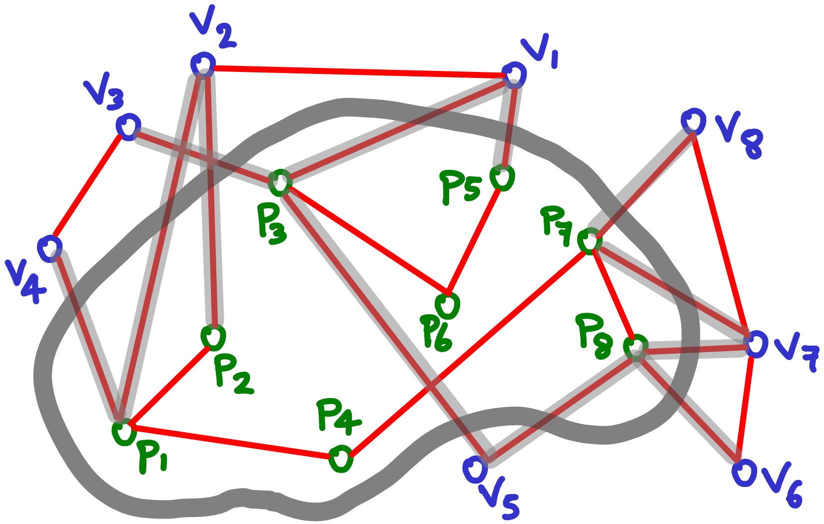


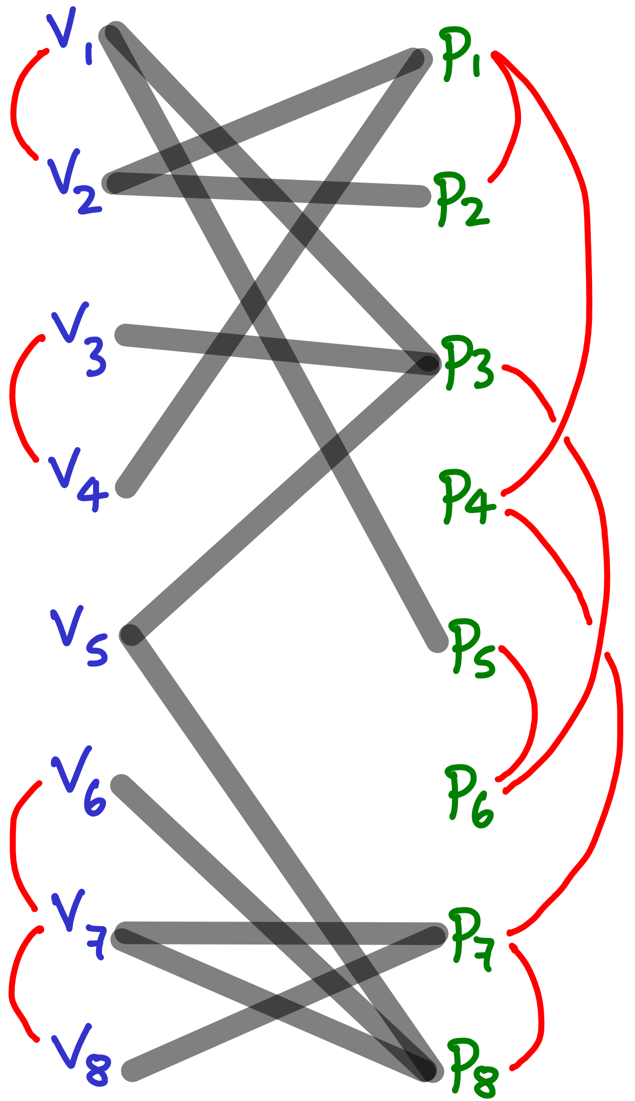


X = #edges that cross a random cut

$$X_e = \begin{cases} 1 & \text{if edge } e \text{ crosses cut} \\ 0 & \text{otherwise} \end{cases}$$

$$X = \sum X_e$$





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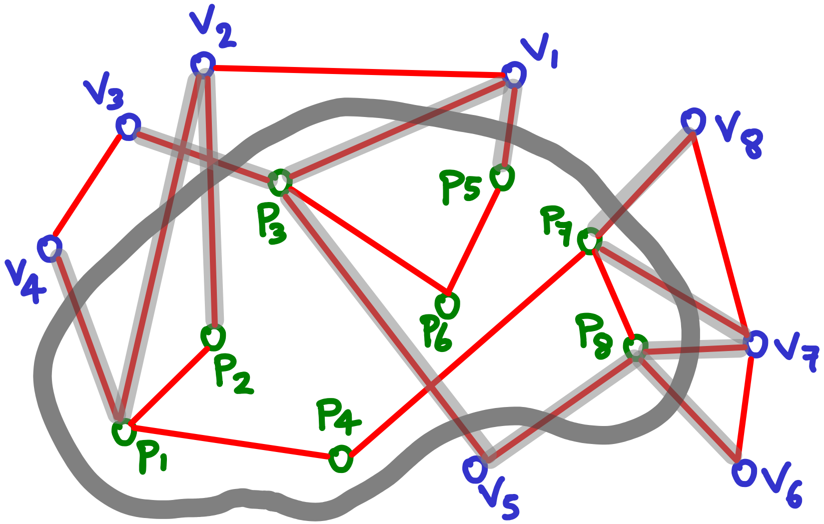

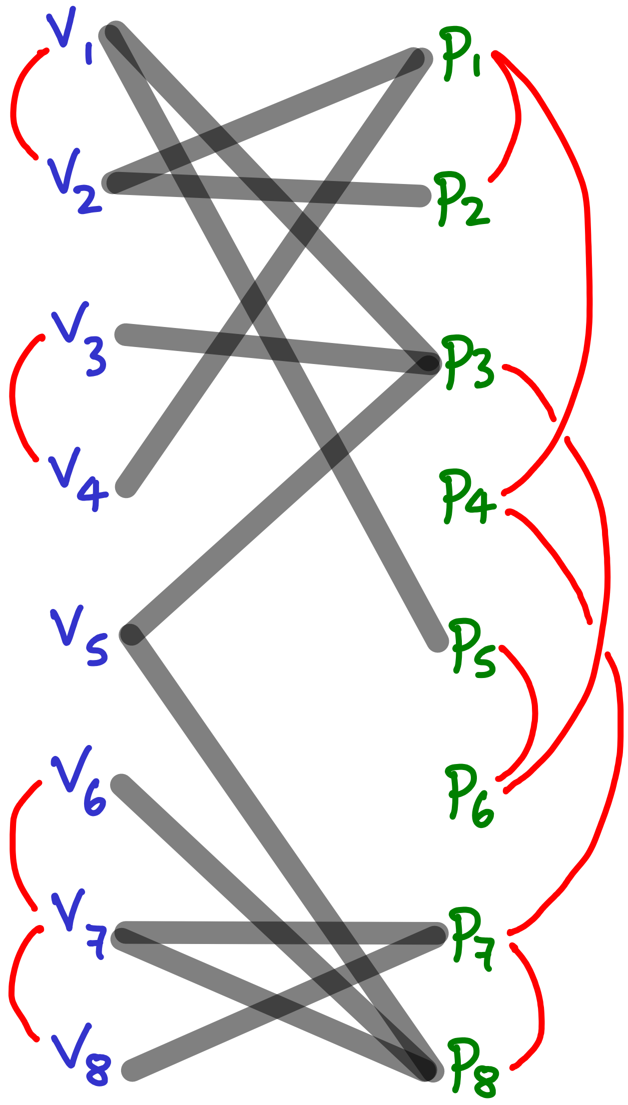

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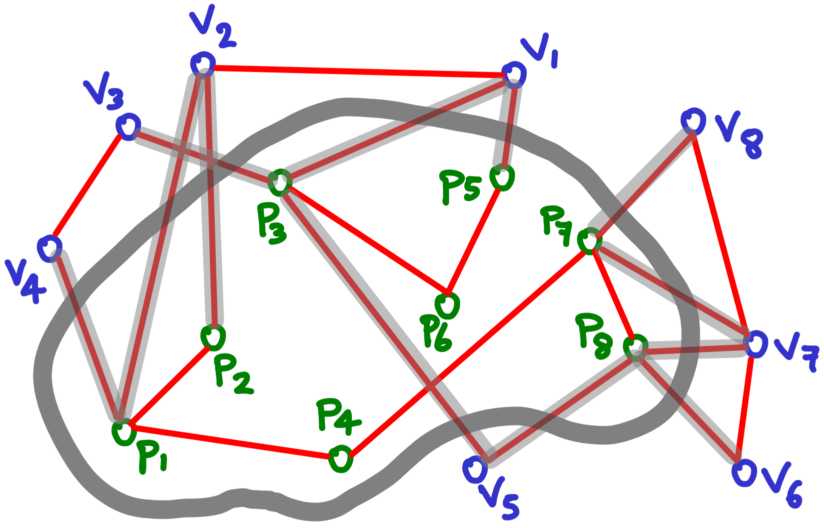

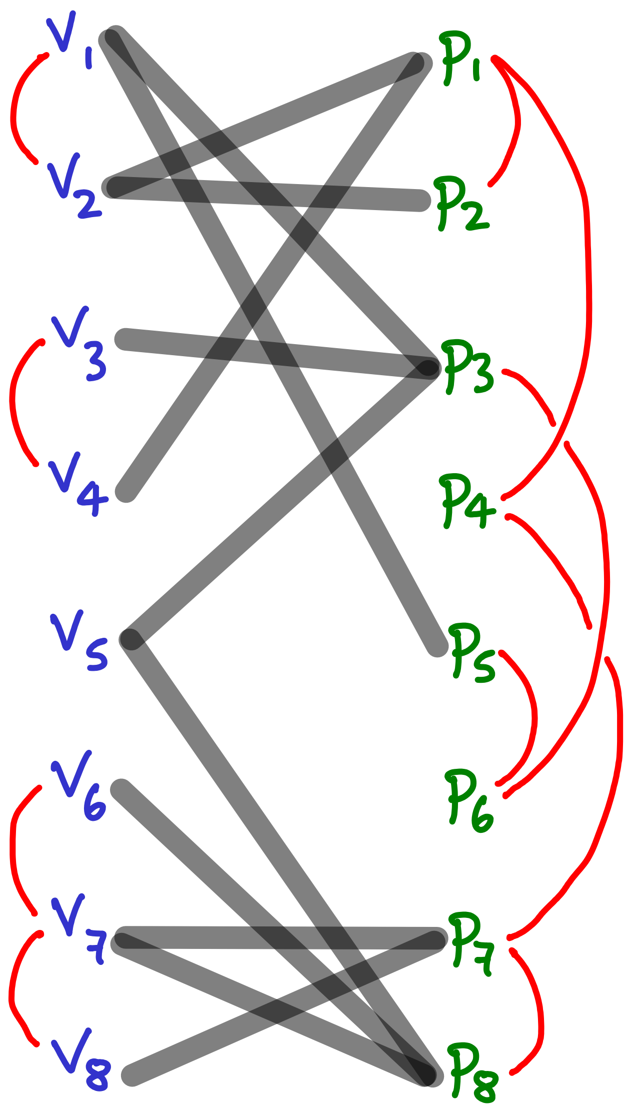

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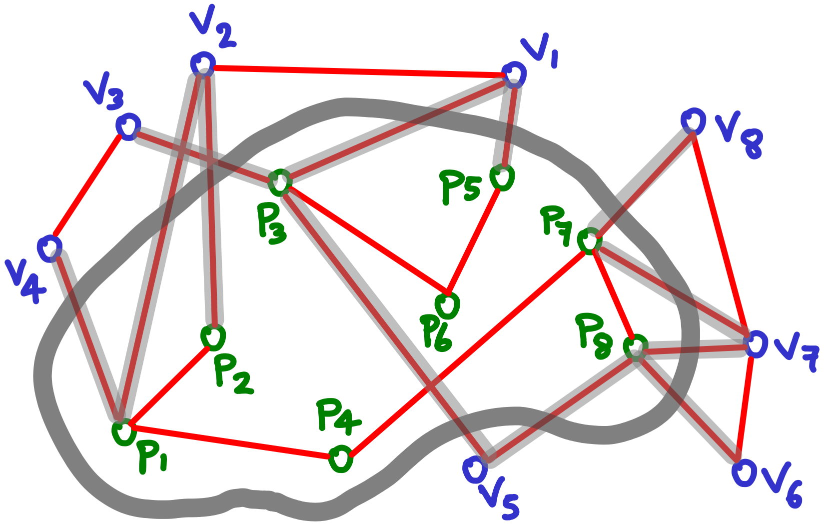
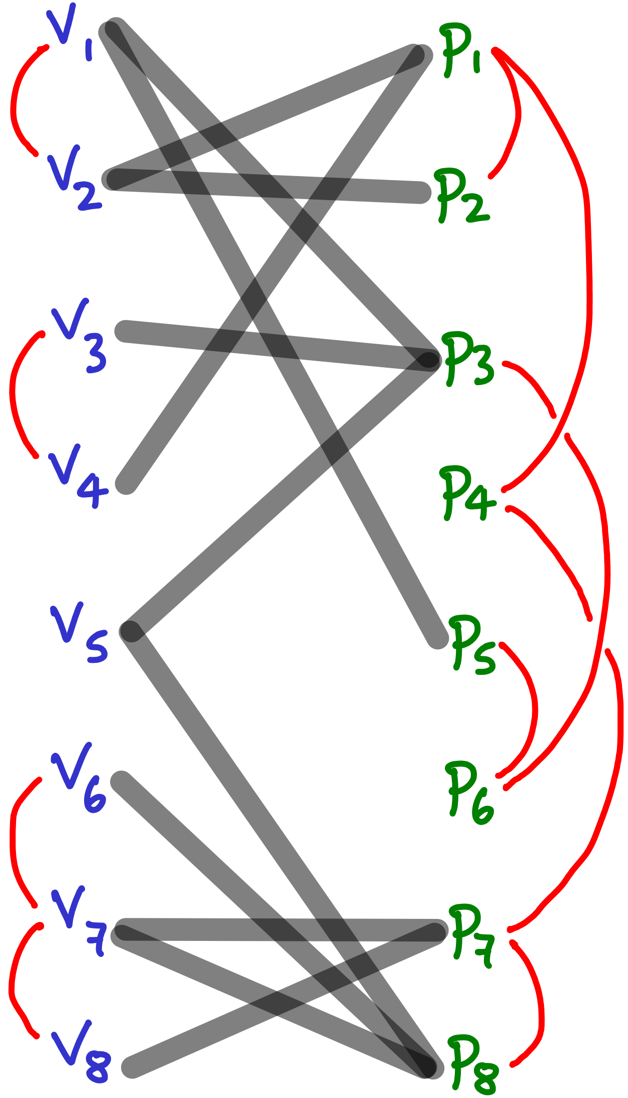
$$\text{Max}\{\text{all possible } X\} \geq E[X]$$





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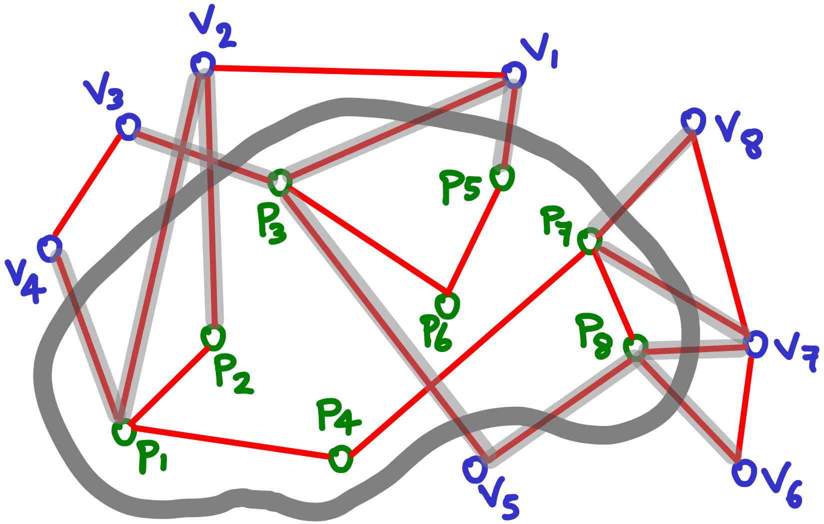
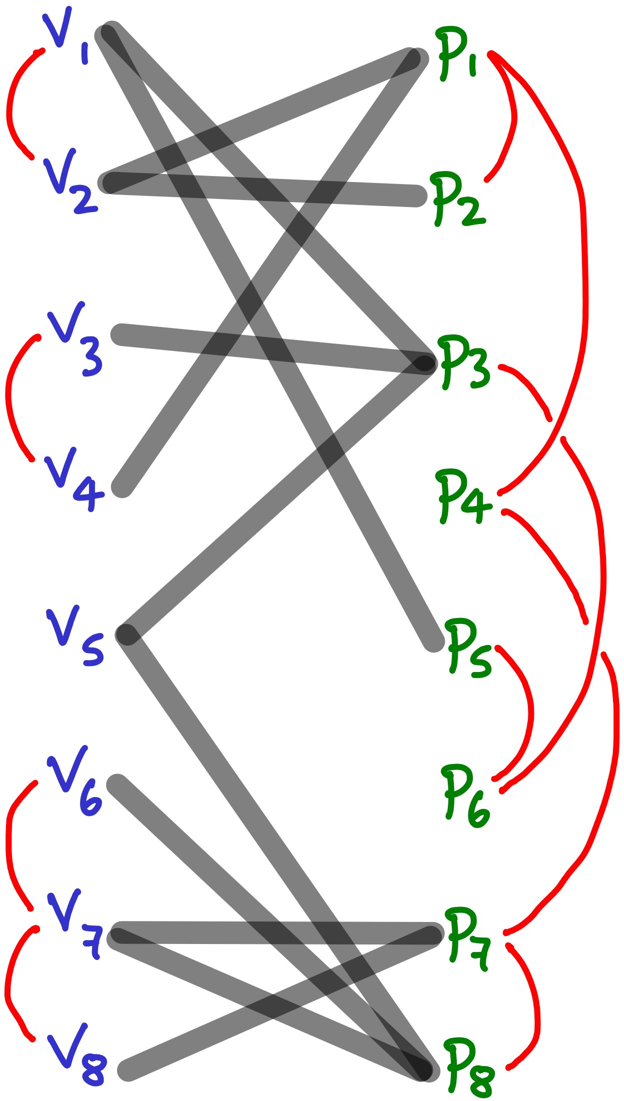
Max{all possible X} $\geq$ E[X] $\Rightarrow$ $\exists$ cut crossing at least half





Matousek-Vondrak

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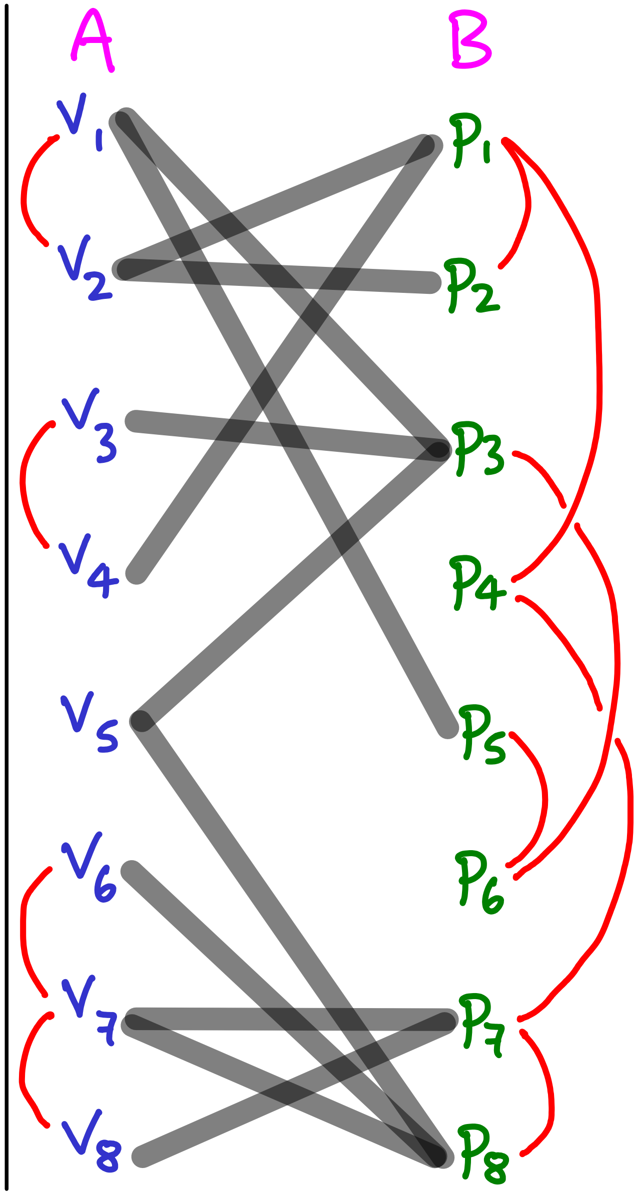

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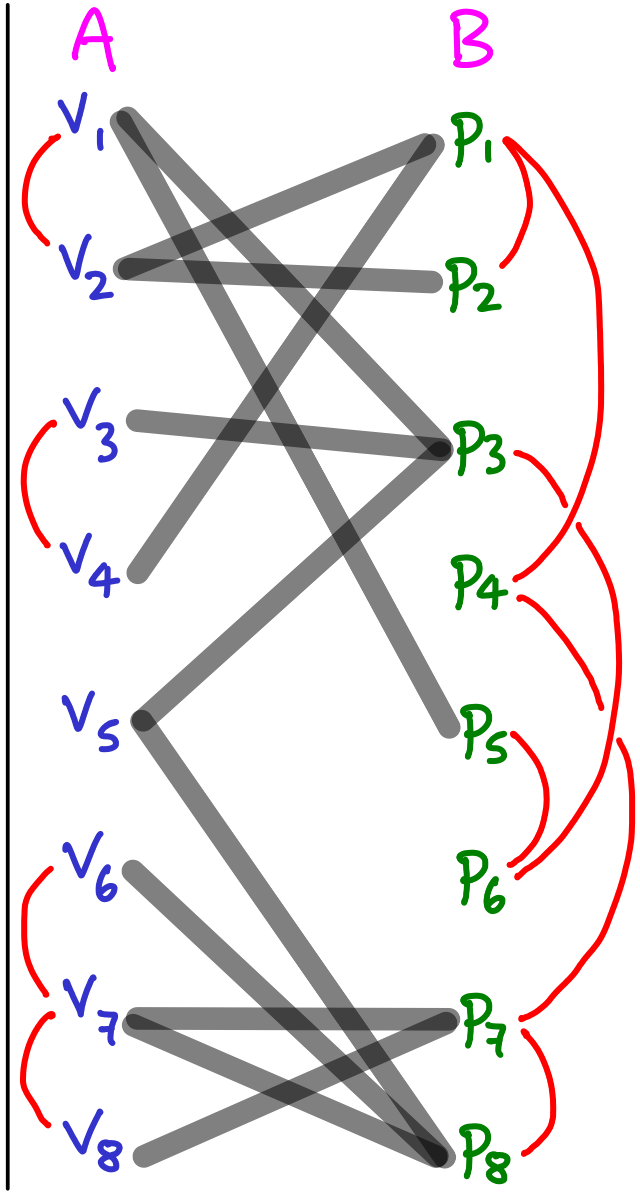

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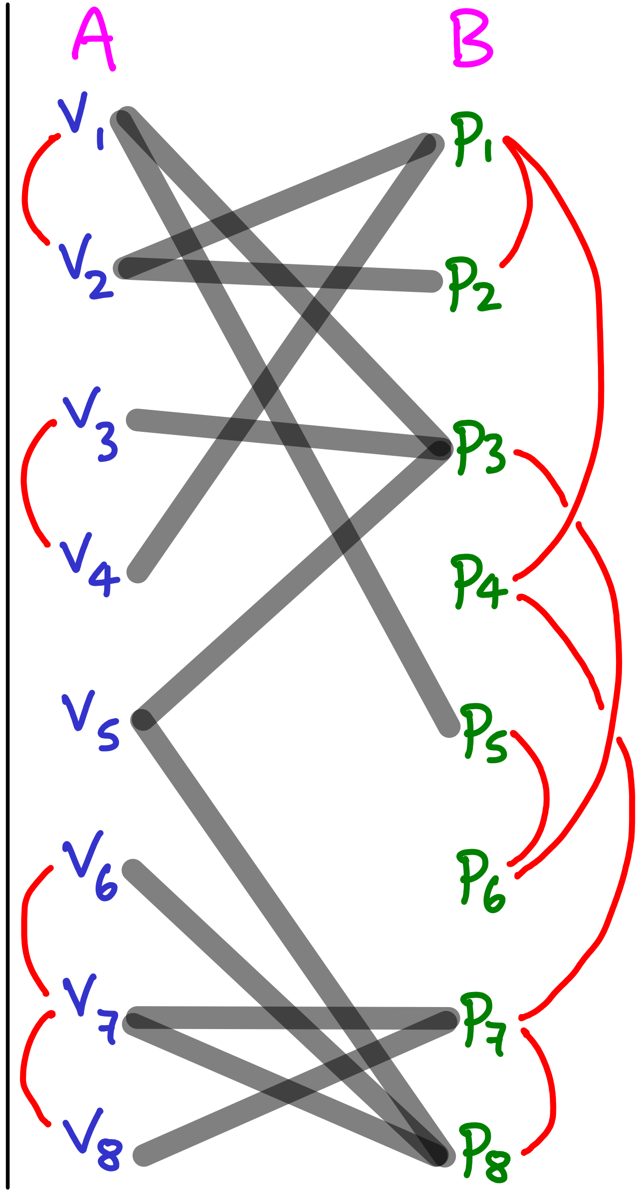

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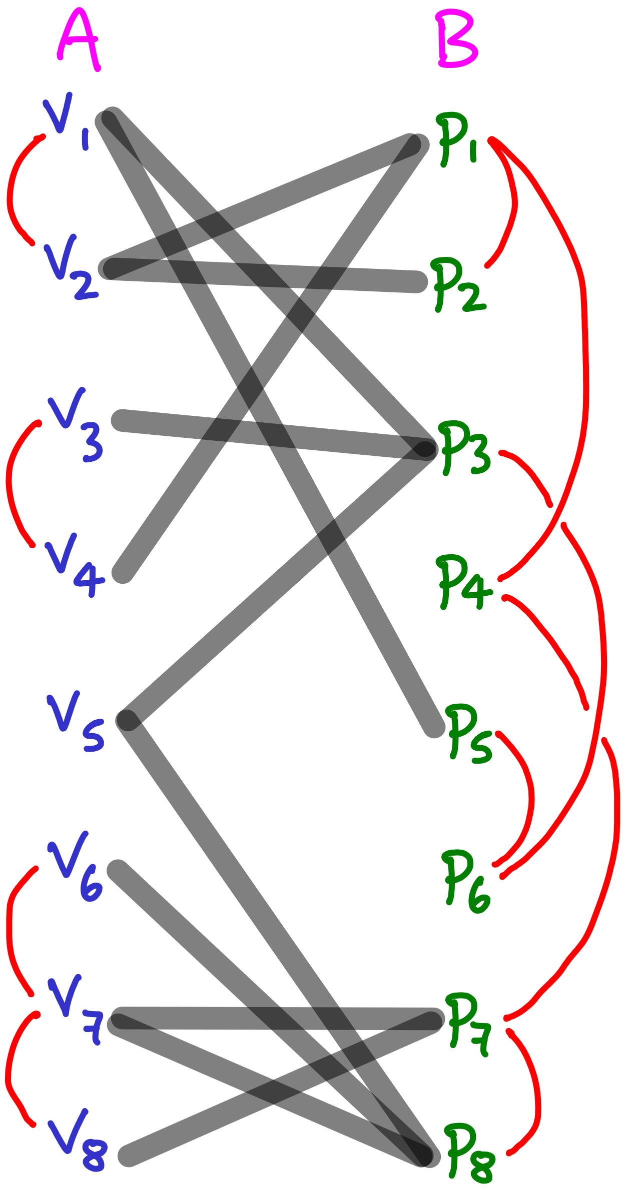
$$P(X_e = 1) = 2\frac{\binom{2n-2}{n-1}}{\binom{2n}{n}}$$



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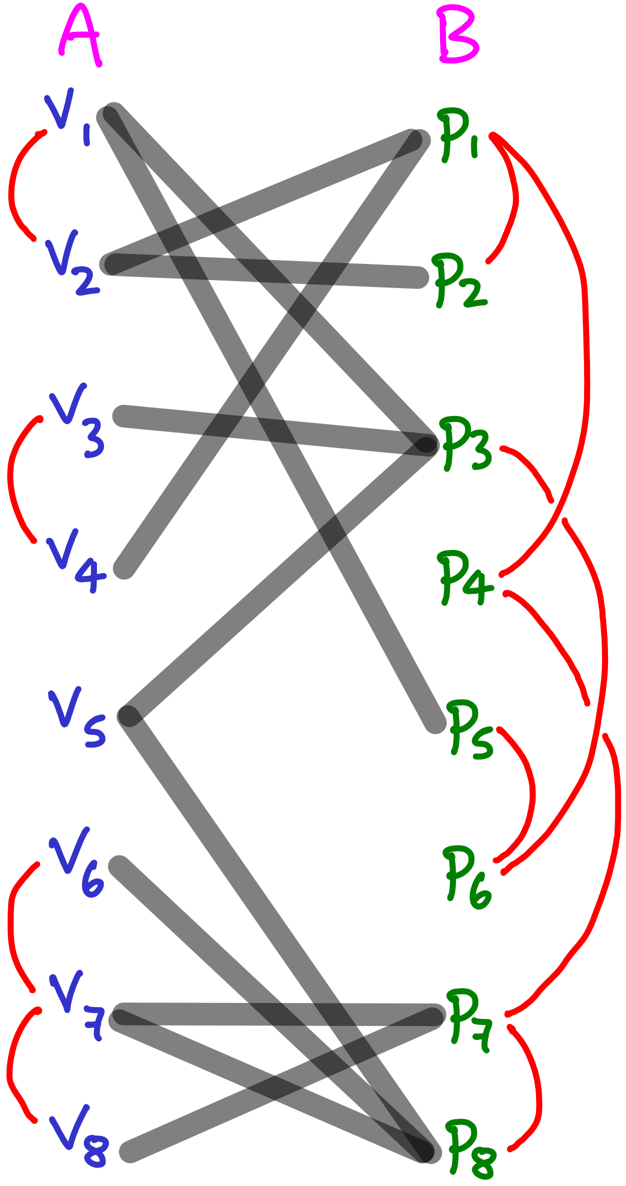

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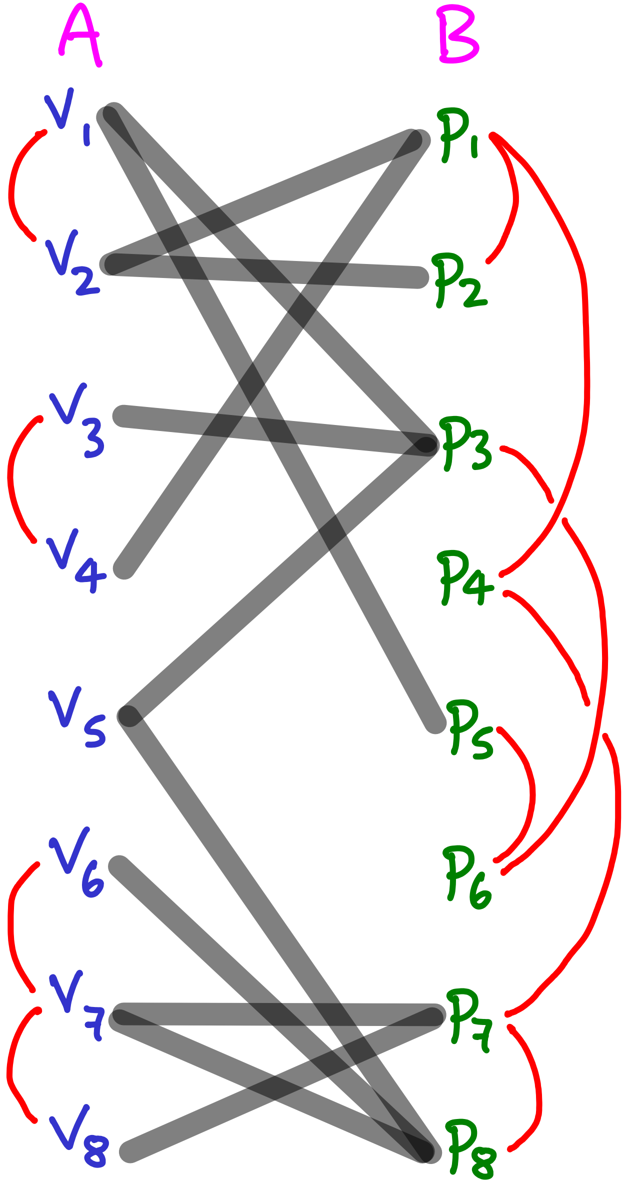

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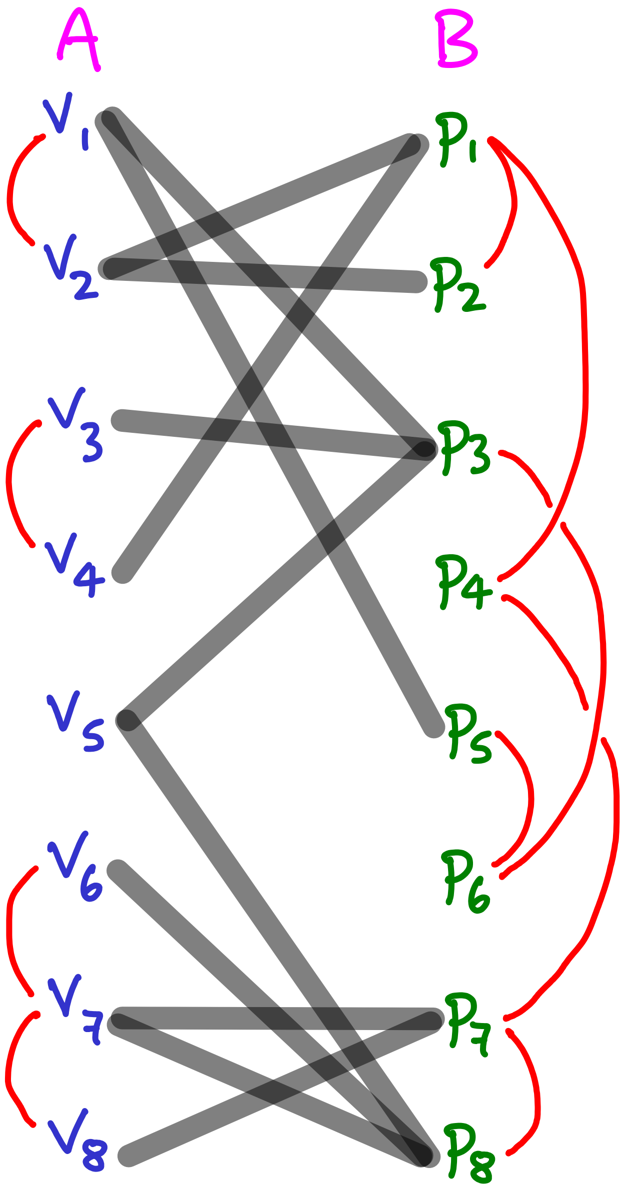

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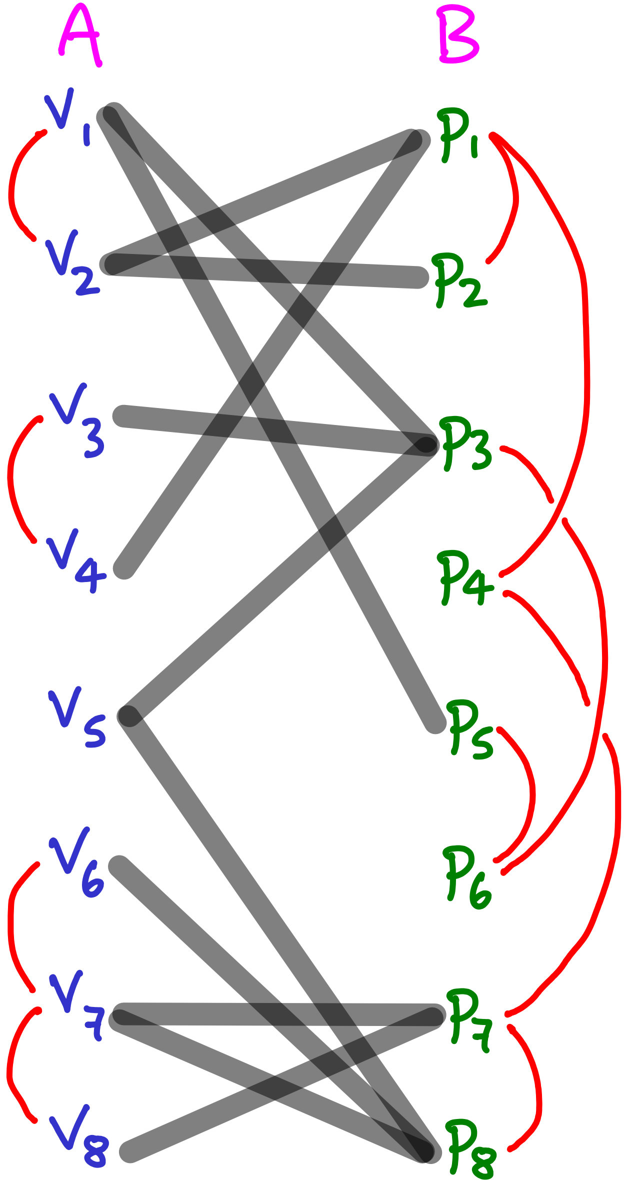

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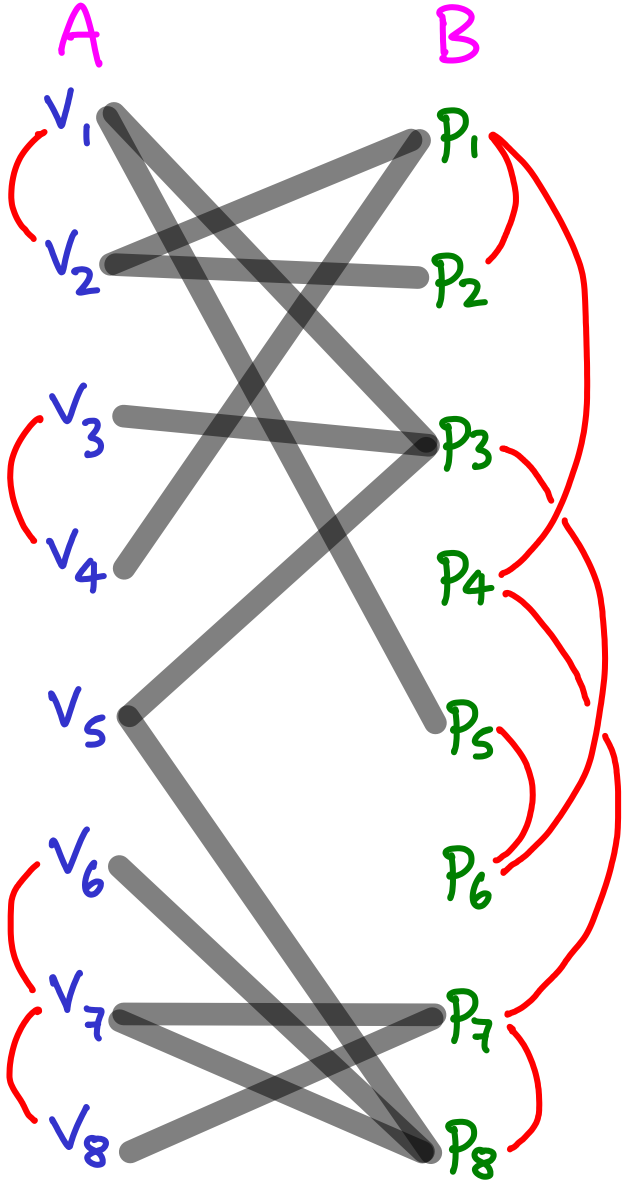
slightly stronger

Matousek-Nesetril



A weak Turán theorem

For $G = \{n, e\}$, if the average degree $d = \frac{2e}{n} \geqslant 1$

then the largest independent set has at least $\frac{n}{2d}$ vertices

$$\frac{n^2}{4e}$$

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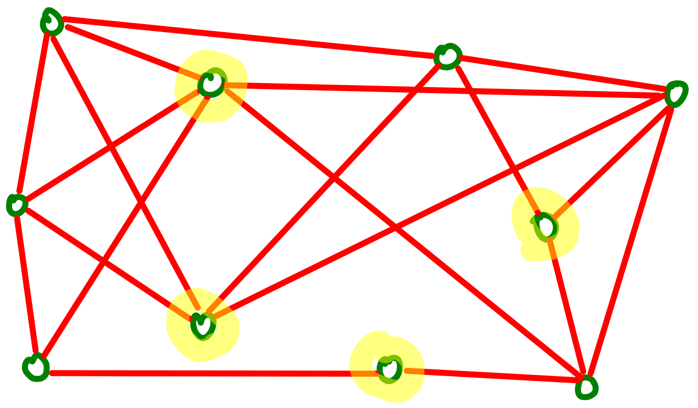

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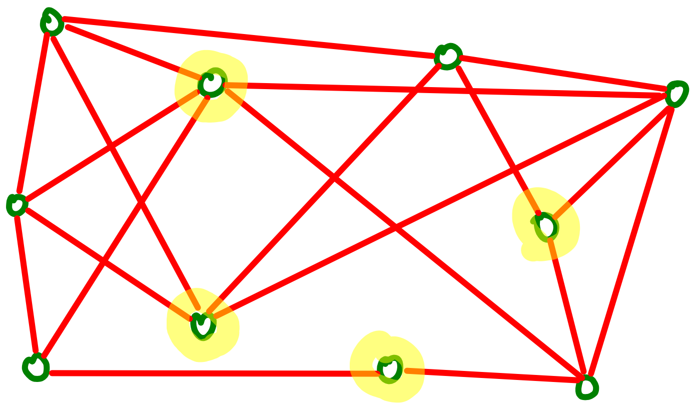

$d=\frac{38}{10}$

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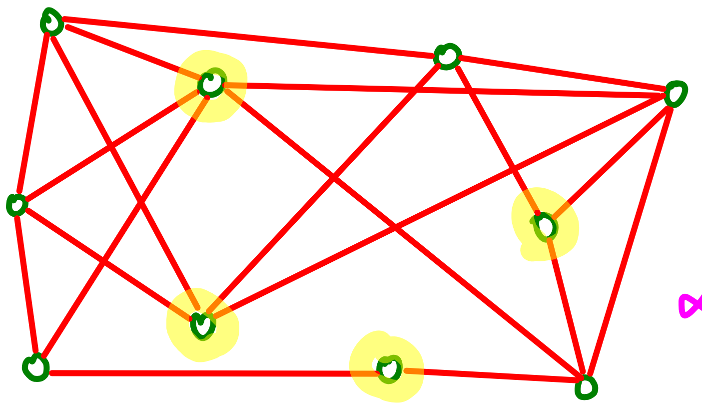

$d = \frac{38}{10}$

$\alpha \geqslant \frac{10}{2 \cdot 3.8} > 1$ ☺

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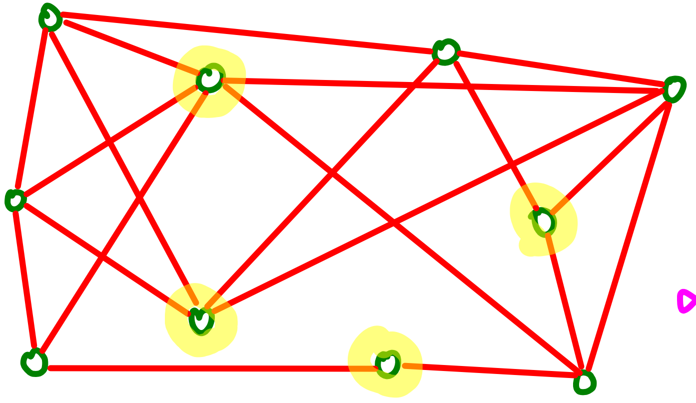
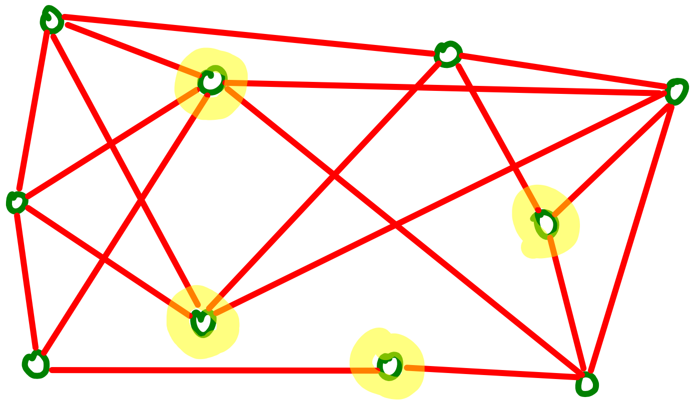


$d=\frac{38}{10}$

$\alpha\geqslant\frac{10}{2\cdot 3.8}>1$:)

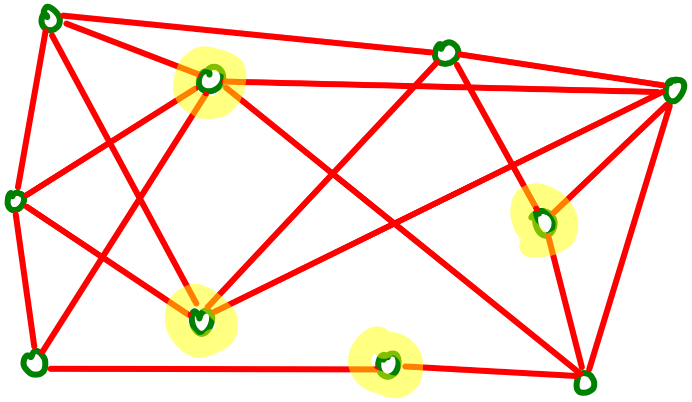
(actual) Turán theorem gives $\geqslant\frac{n}{d+1}$ $\rightarrow \frac{10}{4.8}>2$: in fact $\exists$ graphs matching this weak Turán: For $G = \{n, e\}$, if $d = \frac{2e}{n} \geq 1$ then $\alpha(G) \geq \frac{n}{2d}$



weak Turán: For $G=\{n,e\}$, if $d=\frac{2e}{n}\geq 1$ then $\alpha(G)\geq\frac{n}{2d}$

Proof: • keep each vertex with probability p

• edges survive iff both endpoints do.



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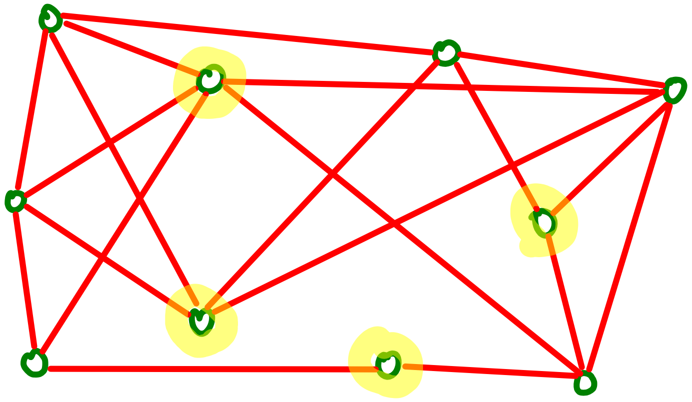
X = # surviving vertices

Y = # surviving edges



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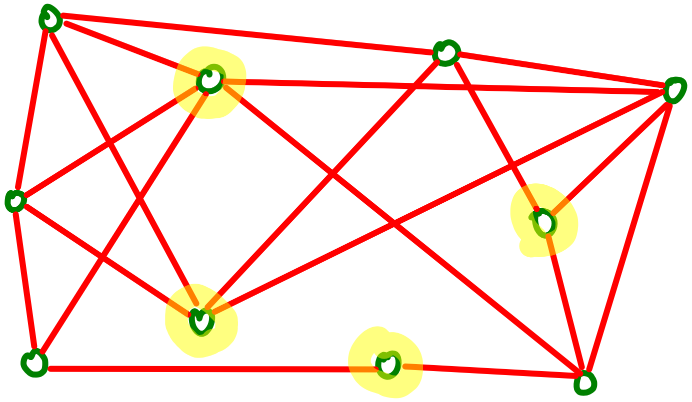
$E[X]=\ ?$ $E[Y]=\ ?$



weak Turán: For $G=\{n,e\}$, if $d=\frac{2e}{n}\geqslant 1$ then $\alpha(G)\geqslant\frac{n}{2d}$

Proof: • keep each vertex with probability p

• edges survive iff both endpoints do.

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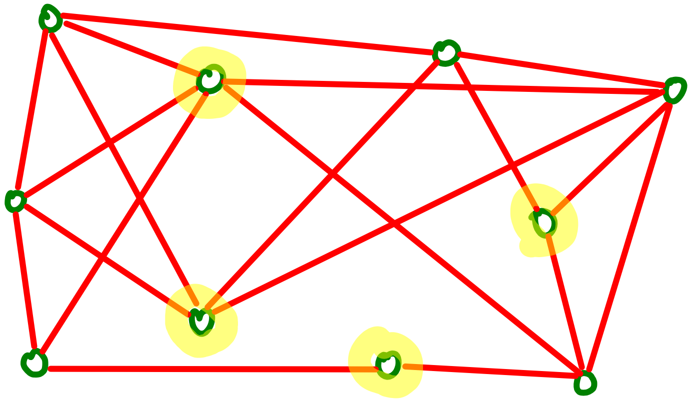
$E[X]=n\cdot p$ $\qquad E[Y]=e\cdot p^2$



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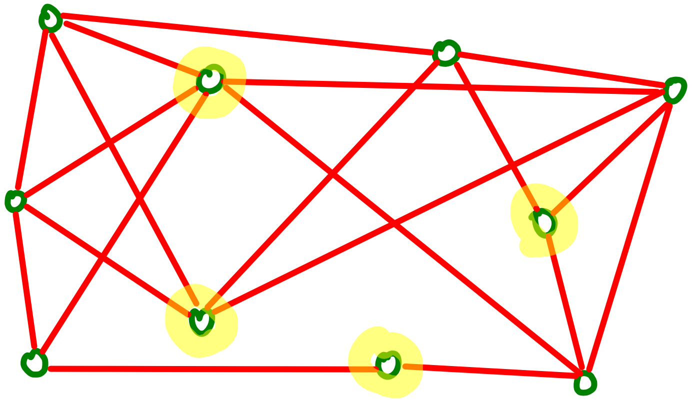

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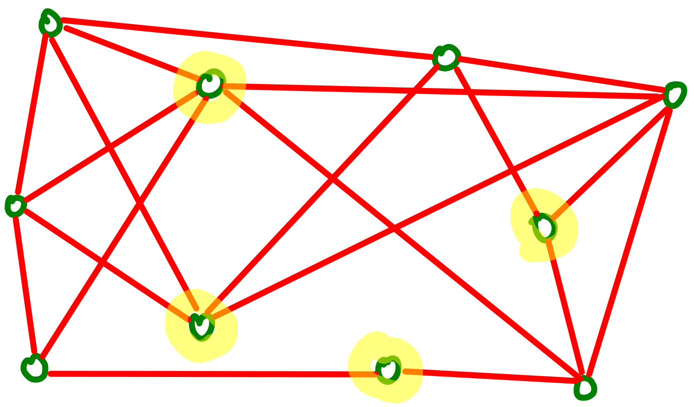

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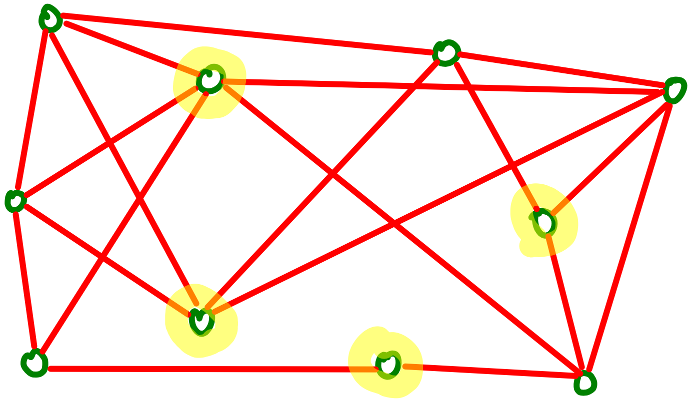

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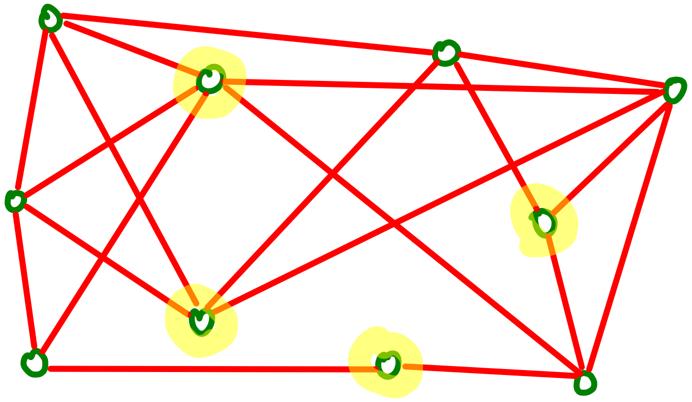
(induced subgraph, according to our rules)



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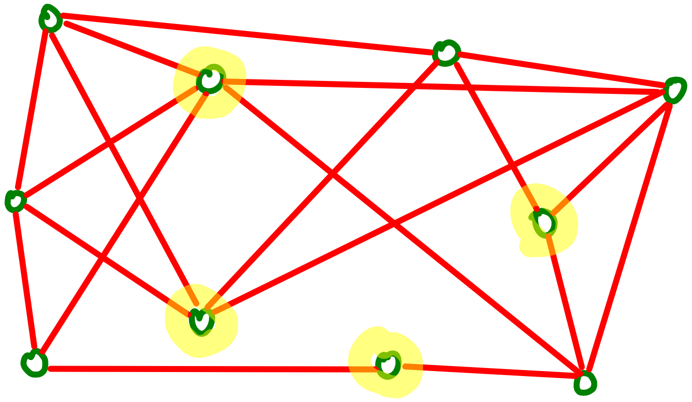

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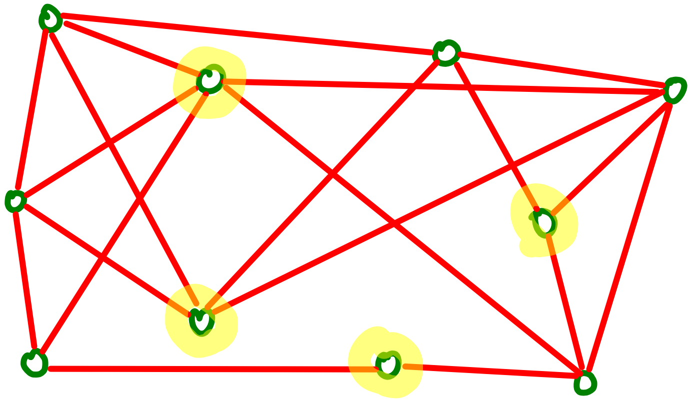

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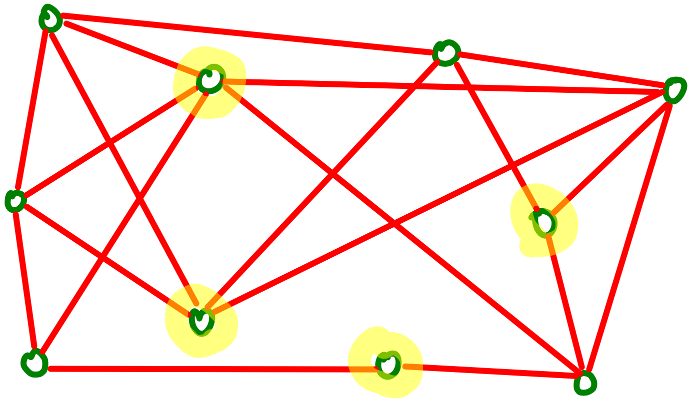

For every edge in S, discard one vertex.

Why?

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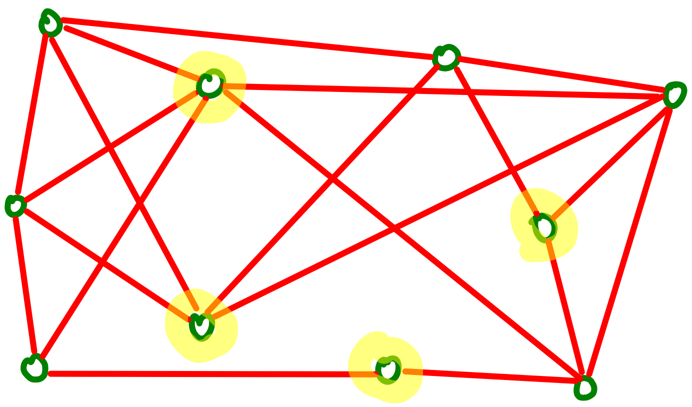


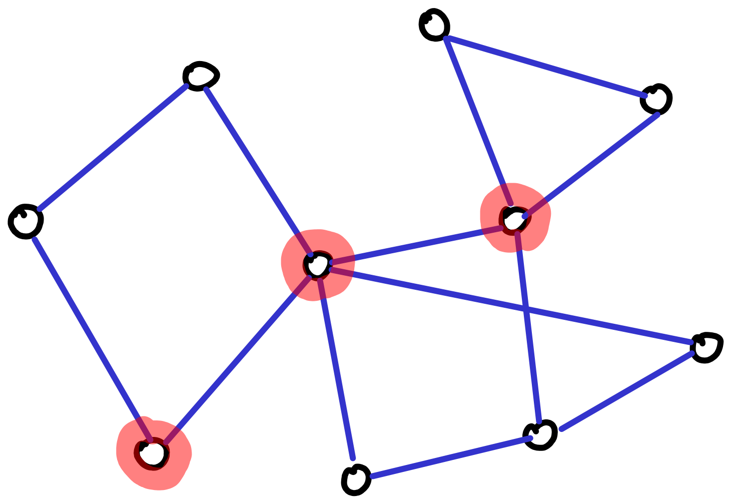
For every edge in S, discard one vertex.

discard $\leq Y$ vertices $\Rightarrow X-Y$ remain.

They form an independent set. $\square$

Dominating Set

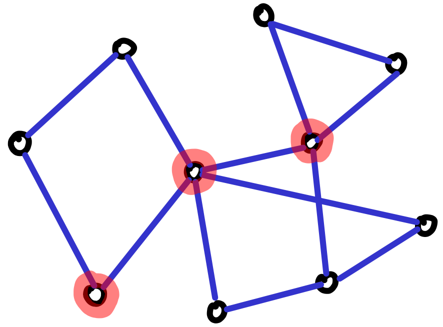
$S \subseteq V$ s.t. every $v \in V$ is either in S or adjacent to a vertex in S



Which towns should get a hospital so every town is adjacent to a hospital?

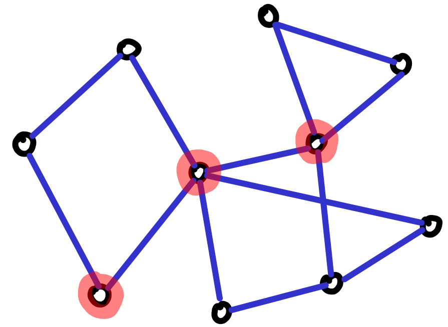
↪ how many hospitals needed?

DOMINATING SET $S \subseteq V$: every $v \in V$ is either in S or adjacent to S



DOMINATING SET $S \subseteq V$: every $v \in V$ is either in S or adjacent to S

$|V| = n$. Suppose min-degree $= \delta > 1$



Then $\exists$ dominating set of size $\leq n \cdot \frac{1+\ln(1+\delta)}{1+\delta}$

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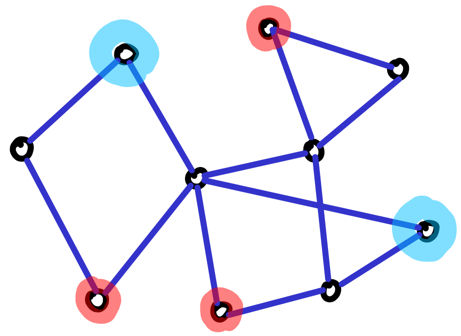

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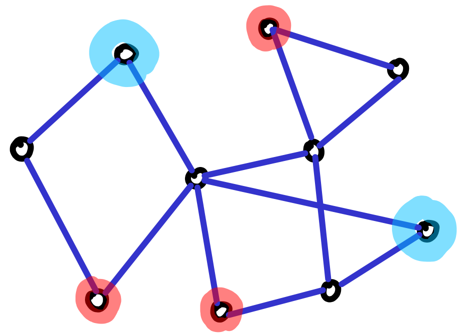

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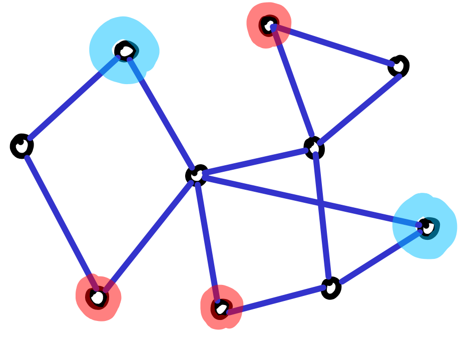
$E[X] = n \cdot p$. $P(v \in Y) = P(v$ & neighbors not in $X) =$?



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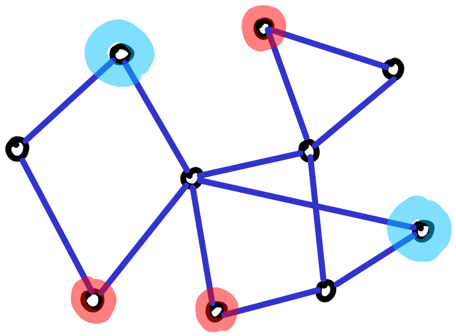

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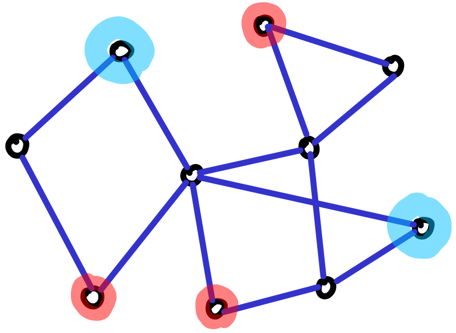

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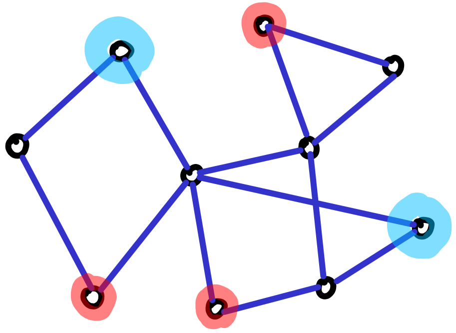

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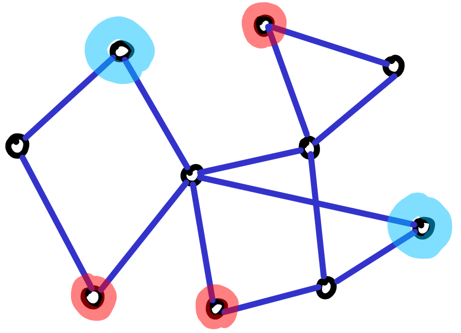

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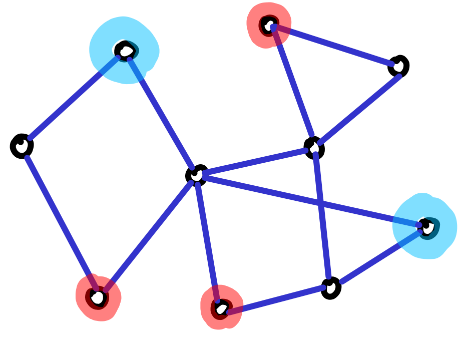
Proof: let every vertex survive with probability p.

$\underline{X = \text{surviving set}}$, $\underline{Y} \subseteq V - X$: $\underline{\text{not neighboring } X}$



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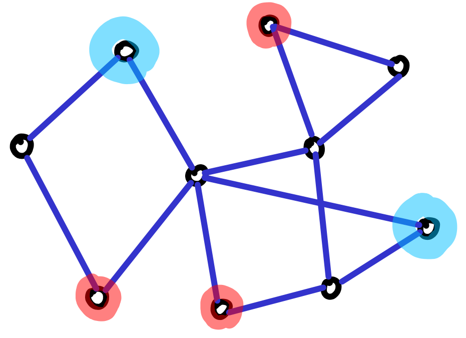

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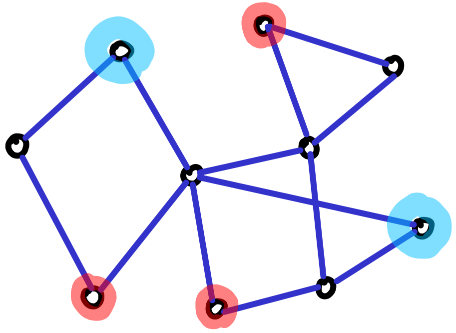

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$E[X+Y] = E[X] + E[Y] = n \cdot p + n \cdot (1-p)^{1+\delta}$ } $\exists$ dominating set at most this size

DOMINATING SET $\leqslant n \cdot p + n \cdot (1-p)^{1+\delta}$

DOMINATING SET $\leq n \cdot p + n \cdot (1-p)^{1+\delta}$

$$\leq n \cdot p + n e^{-p(1+\delta)}$$

(for $p \geq 0$)

$1-p \leq e^{-p} \Rightarrow$

Taylor: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^k}{k!} + \cdots$

DOMINATING SET $\leq n \cdot p + n \cdot (1-p)^{1+\delta}$

$$\leq \underbrace{n \cdot p + ne^{-p(1+\delta)}}_{n \cdot \frac{\ln(1+\delta)}{1+\delta} + ne^{-\ln(1+\delta)}}$$

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$$n \cdot \frac{\ln(1+\delta)}{1+\delta} + n e^{-\ln(1+\delta)}$$

$$= n \cdot \frac{\ln(1+\delta)}{1+\delta} + n \frac{1}{1+\delta}$$

$$= n \frac{1+\ln(1+\delta)}{1+\delta} \qquad \square$$

DOMINATING SET $\leq n \cdot p + n \cdot (1-p)^{1+\delta}$

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$$= n \frac{1 + \ln(1+\delta)}{1+\delta} \qquad \square$$

e.g. $\delta = 9 \Rightarrow$ dominating set $\leq \frac{n}{3}$

$$\sim n \cdot \frac{1+2.3}{10}$$

HYPERGRAPHS

$H = \{V, E\}$ V: vertex set E: edges ~ hyperedges

An edge is a subset of V.

Hypergraphs

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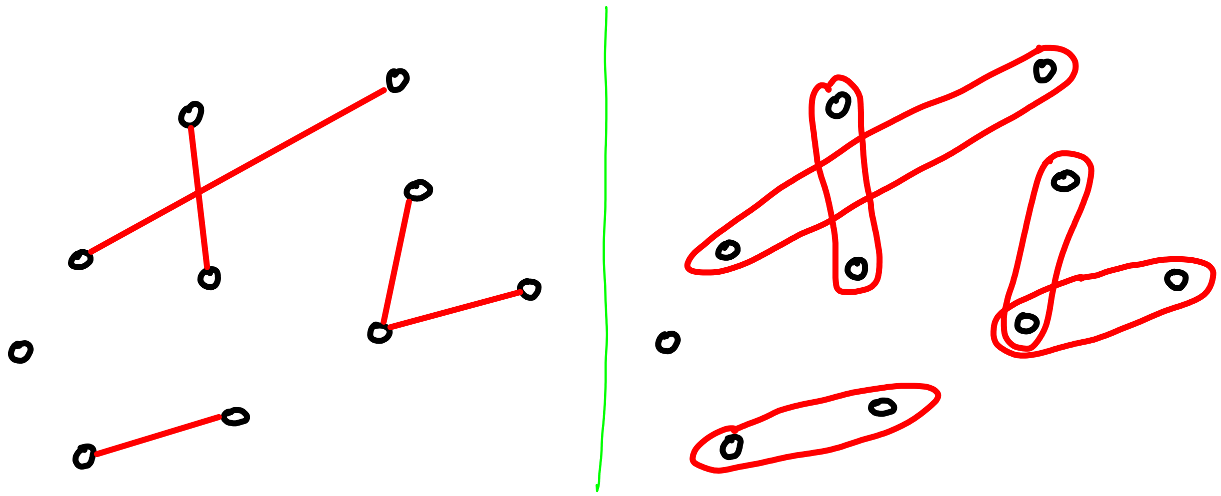
An edge is a subset of V.

In classic graphs each such subset has size 2

↳ 2-uniform hypergraph



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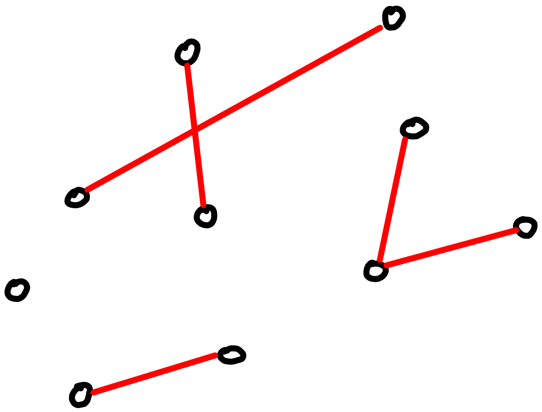
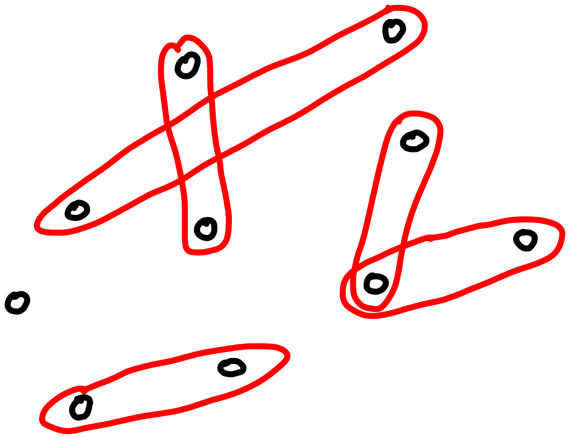
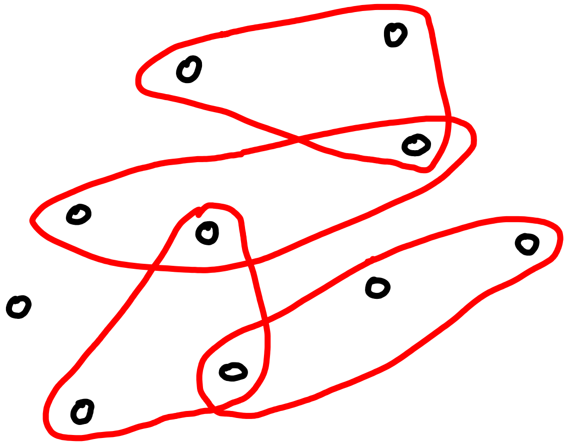


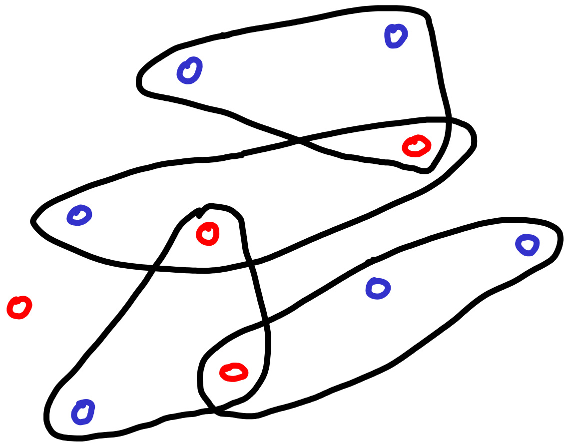


n-uniform hypergraph:
every edge has n vertices

$n=3$



2-colorable hypergraph : vertices can be colored red or blue,
s.t. no edge is monochromatic



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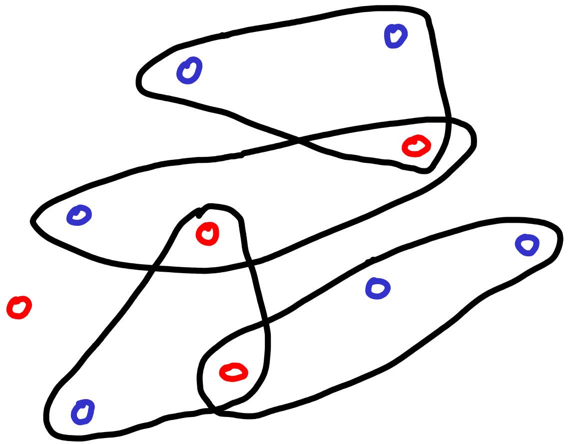
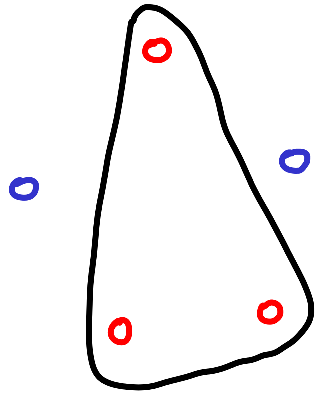


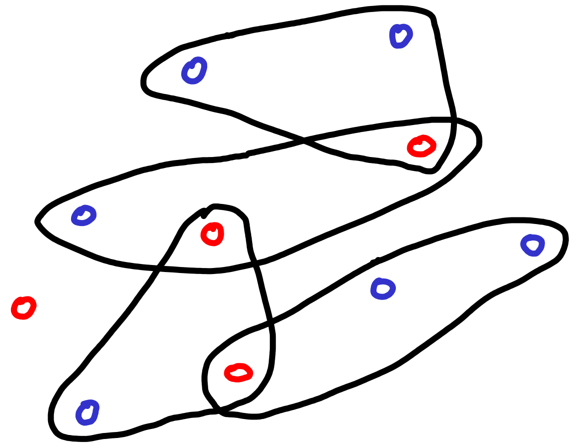


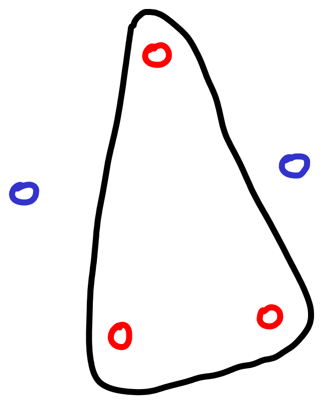
V=5

Complete 3-uniform

Not 2-colorable

2-colorable hypergraph: vertices can be colored red or blue, s.t. no edge is monochromatic





$V=5$

Complete 3-uniform

Not 2-colorable

Claim: if n-uniform with $< 2^{n-1}$ edges, then 2-colorable

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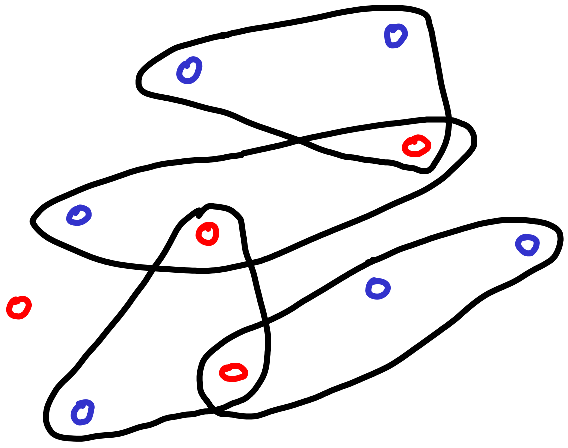

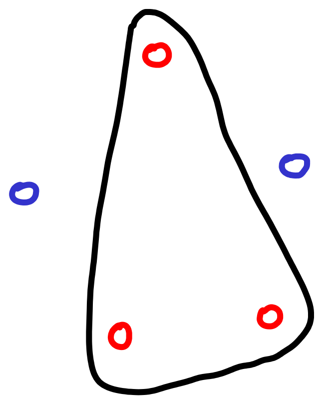

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This is a lower bound on # edges required to get a non-2-colorable n-uniform hypergraph

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See Alon-Spencer, p.7 for upper bound & more refined lower bound