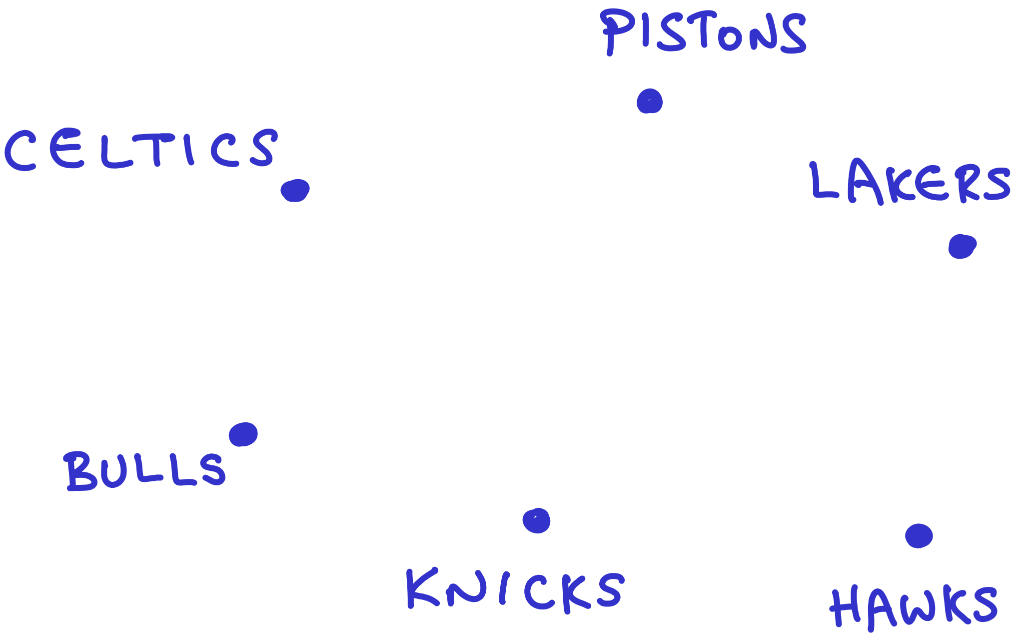
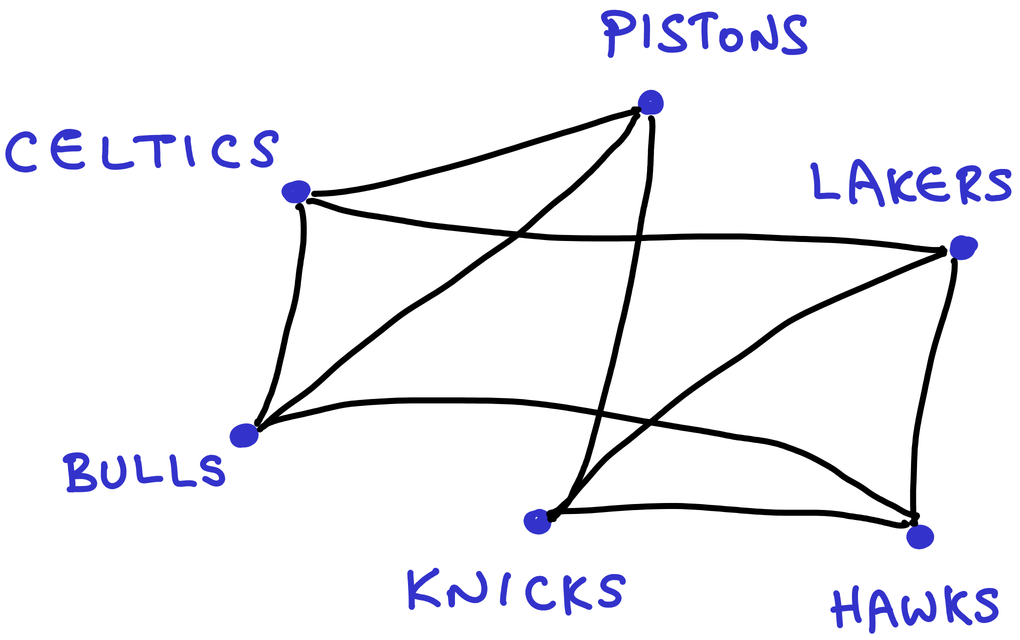
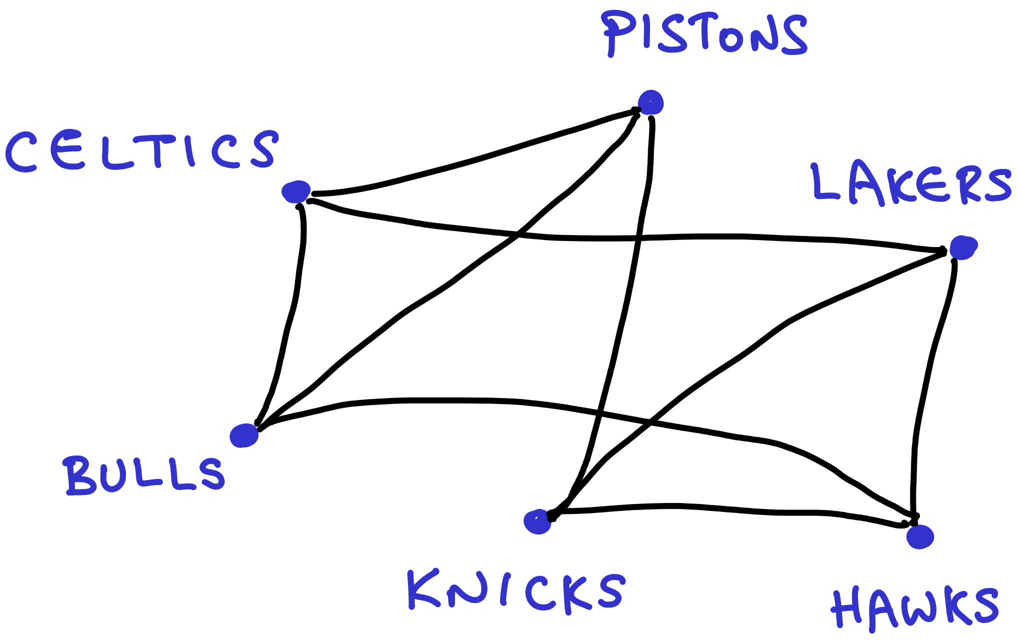
6 teams





6 teams

each plays 3 others



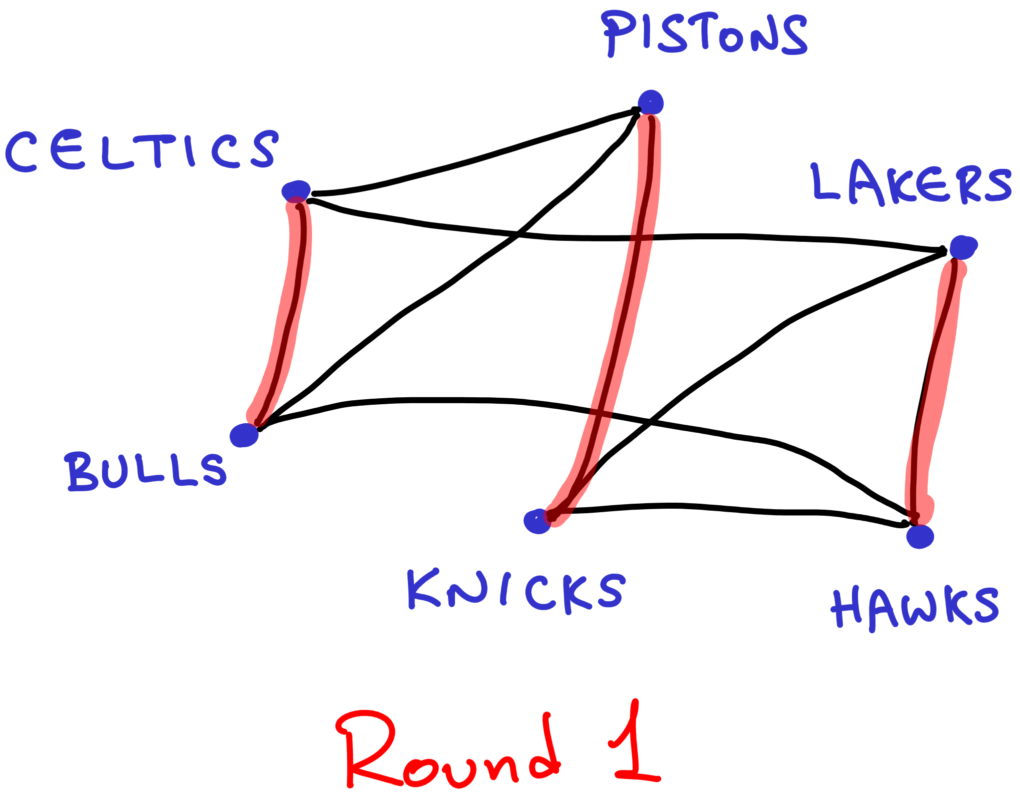
6 teams

each plays 3 others

Games can be played in parallel

Want to finish ASAP

How many rounds required?



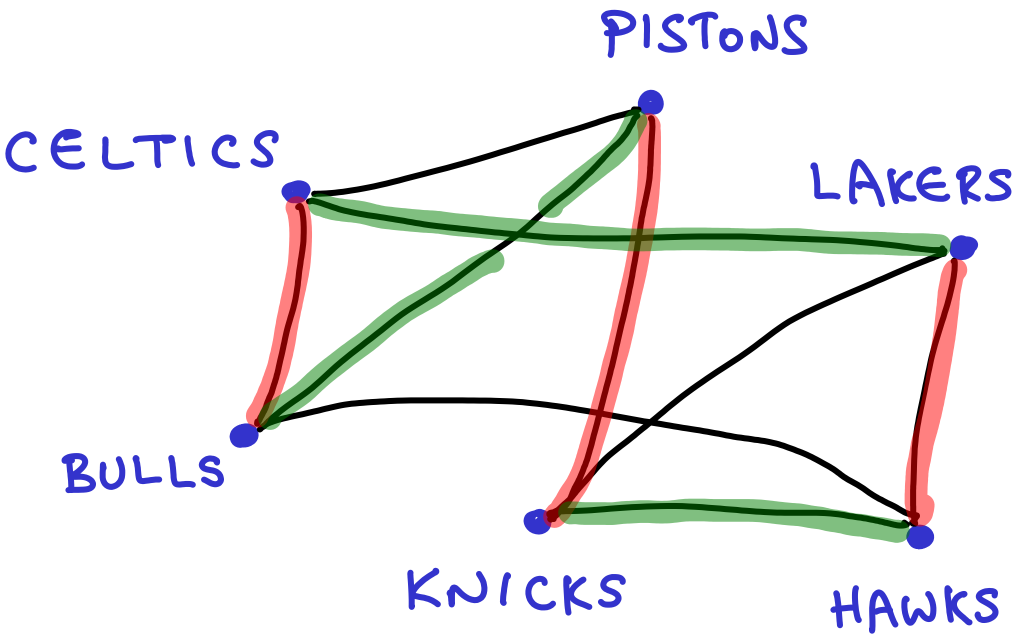
6 teams

each plays 3 others

Games can be played in parallel

Want to finish ASAP

How many rounds required?



Round 1
Round 2

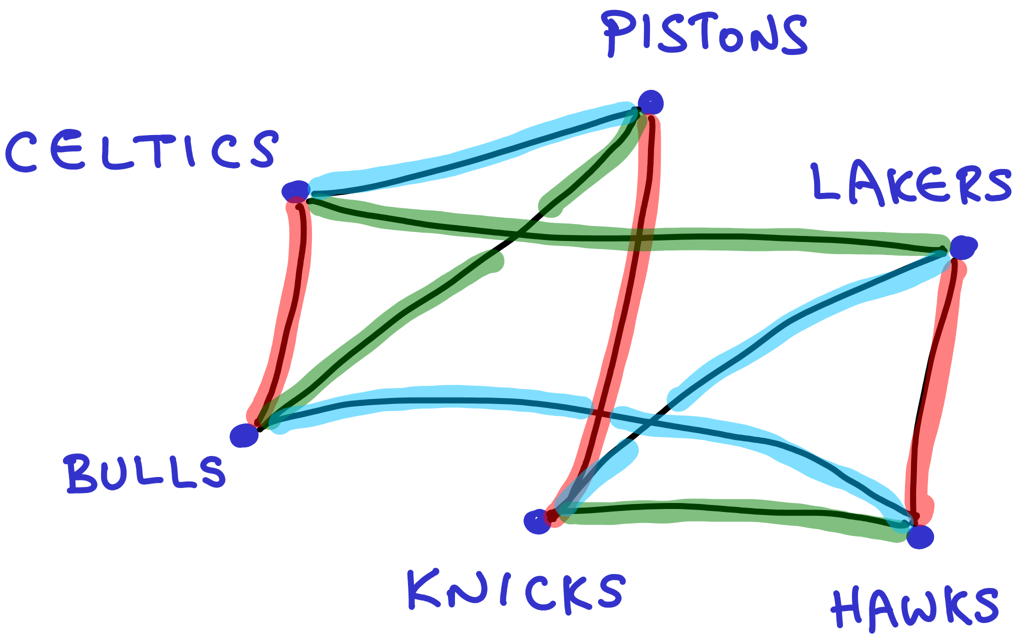
6 teams

each plays 3 others

Games can be played in parallel

Want to finish ASAP

How many rounds required?



Round 1
Round 2
Round 3

6 teams

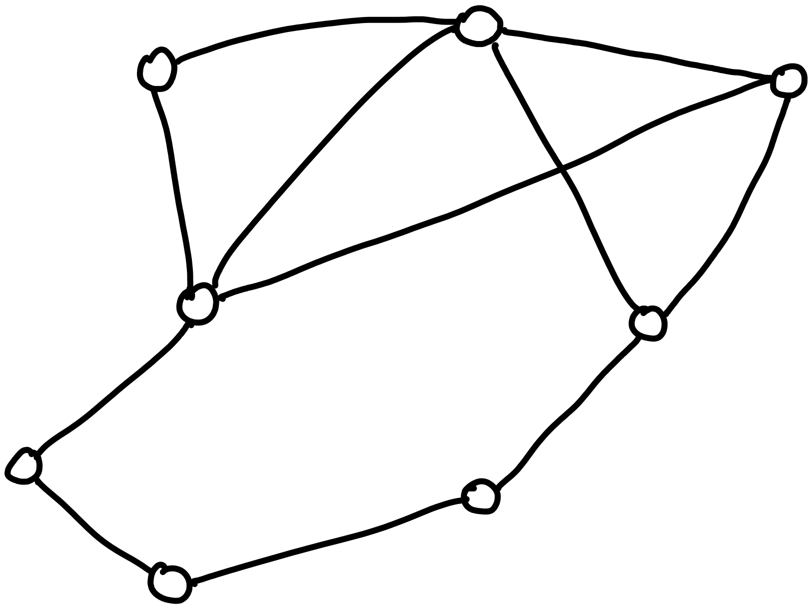
each plays 3 others

Games can be played in parallel

Want to finish ASAP

How many rounds required?

EDGE COLORING - no adjacent edges w same color



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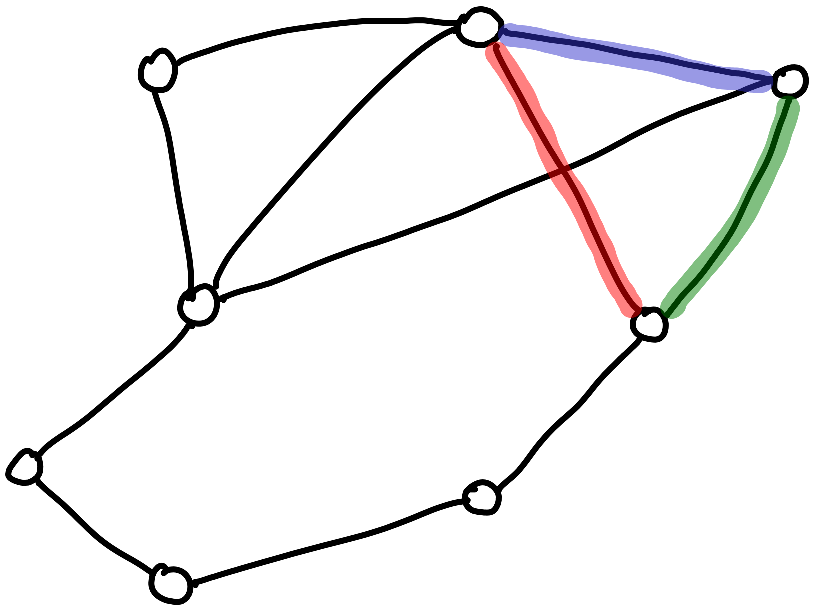
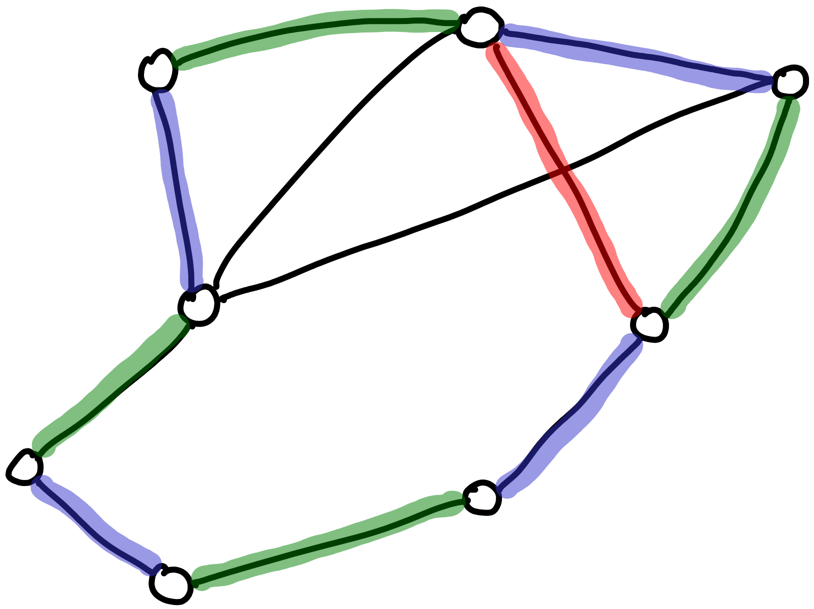
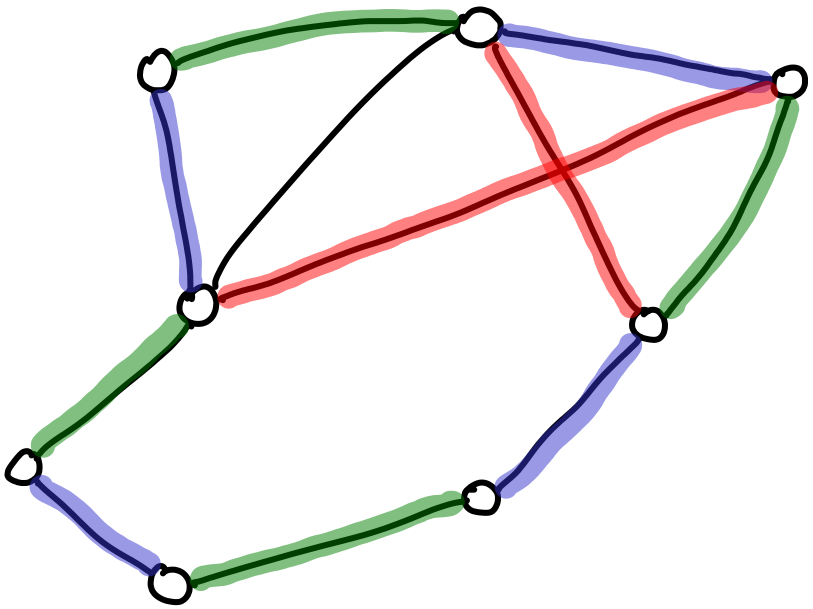


EDGE COLORING - no adjacent edges w same color

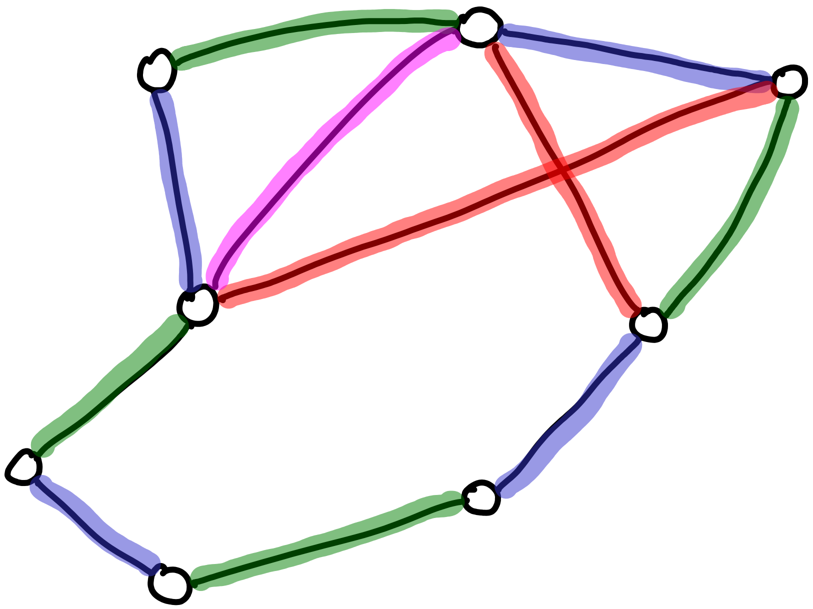


EDGE COLORING - no adjacent edges w same color



EDGE COLORING - no adjacent edges w same color

$\chi'(G)$: min # colors : edge-chromatic #

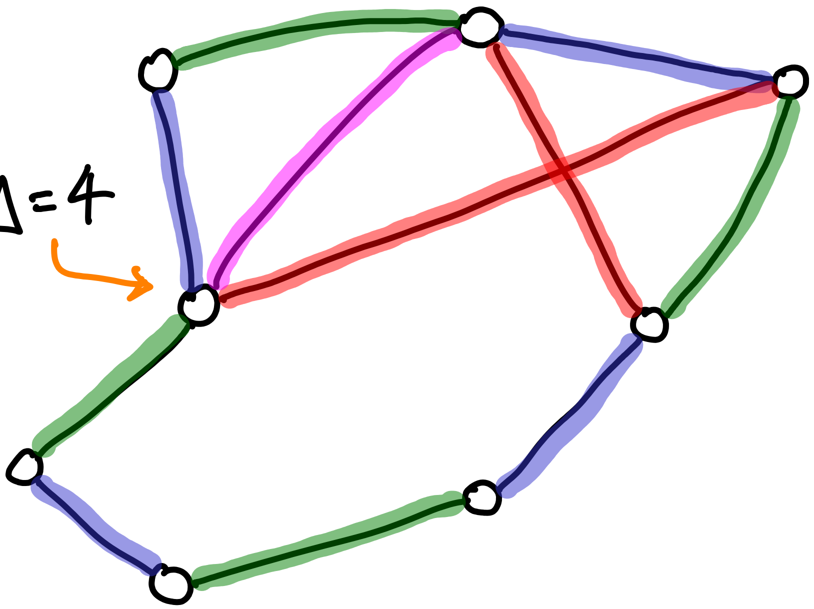


$\chi' \leq 4$ can we do better?

EDGE COLORING - no adjacent edges w same color

$\chi'(G)$: min # colors : edge-chromatic #

$\Delta = 4$

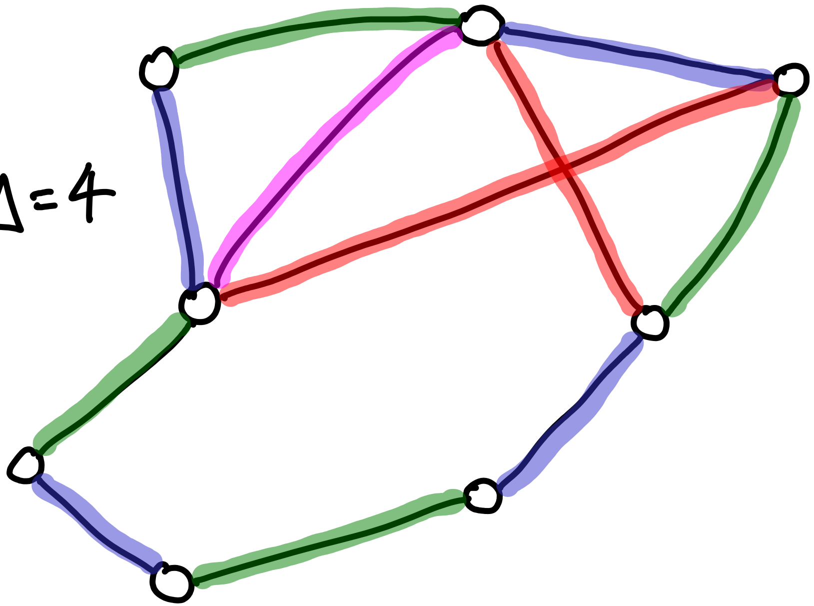


$\chi' = 4$
$\geq \Delta$ {max degree}

EDGE COLORING - no adjacent edges w same color

$\chi'(G)$: min # colors : edge-chromatic #

$\Delta = 4$



$\chi' = 4$

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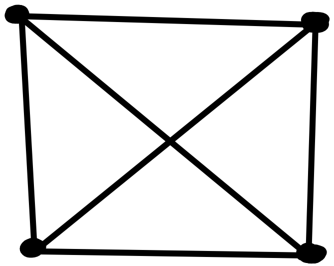
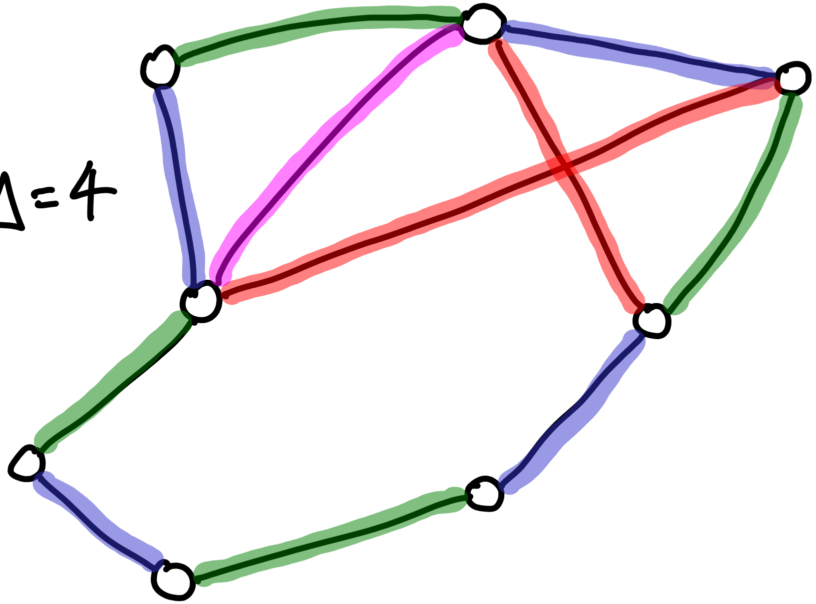


$\chi' = ?$

EDGE COLORING - no adjacent edges w same color

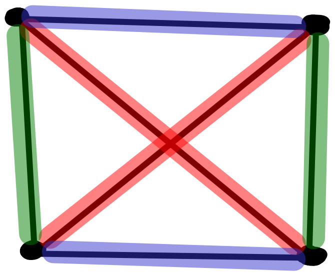
$\chi'(G)$: min # colors : edge-chromatic #

$\Delta = 4$



$\chi' = 4$

$\geq \Delta$ {max degree}



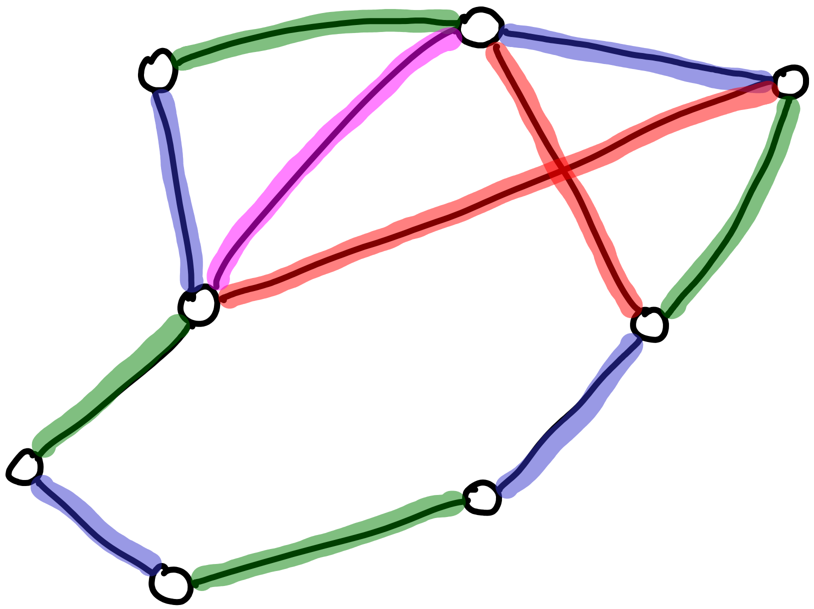
K_4

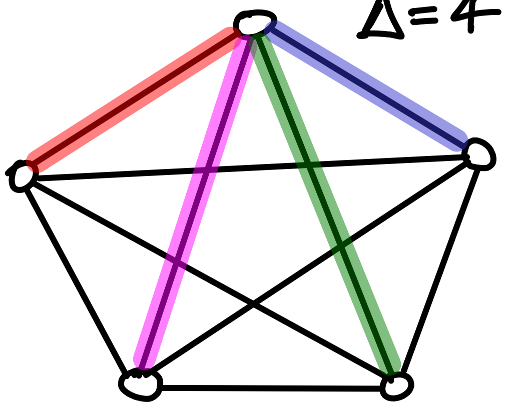
$\Delta = 3$

$\chi' = 3$

EDGE COLORING - no adjacent edges w same color

$\chi'(G)$: min # colors : edge-chromatic #





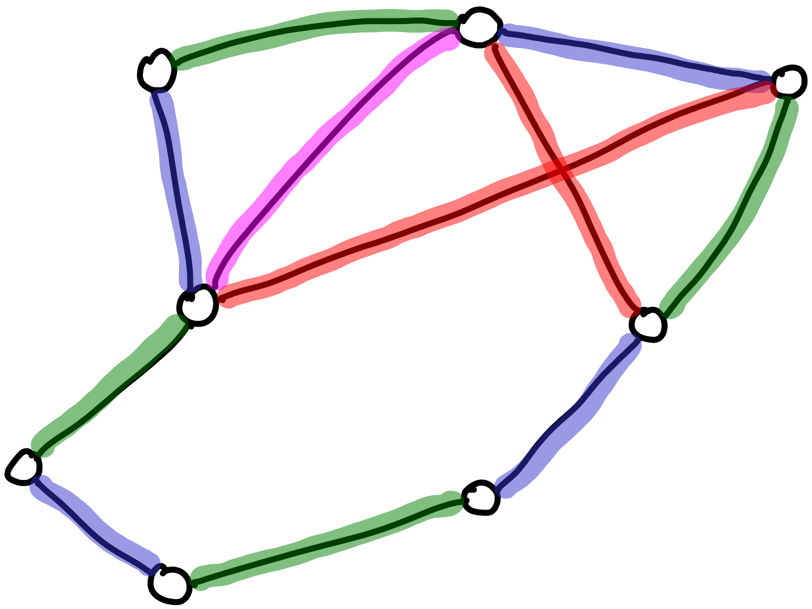
$\Delta = 4$

K_5

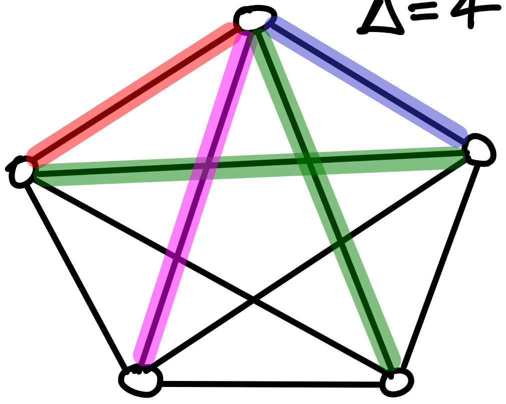
$\chi' = 4 \geq \Delta$

EDGE COLORING - no adjacent edges w same color

$\chi'(G)$: min # colors : edge-chromatic #



$\Delta = 4$



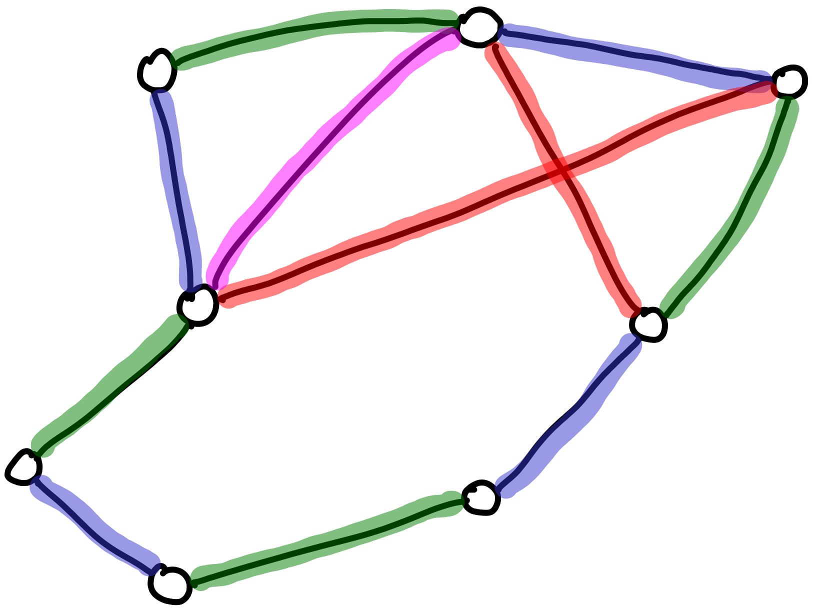
WLOG ● avoiding new color

$\chi' = 4$

$\geq \Delta$

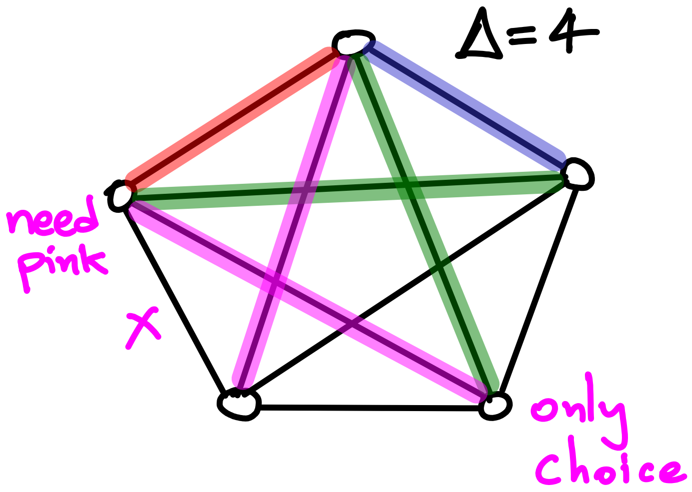
EDGE COLORING - no adjacent edges w same color

$\chi'(G)$: min # colors : edge-chromatic #



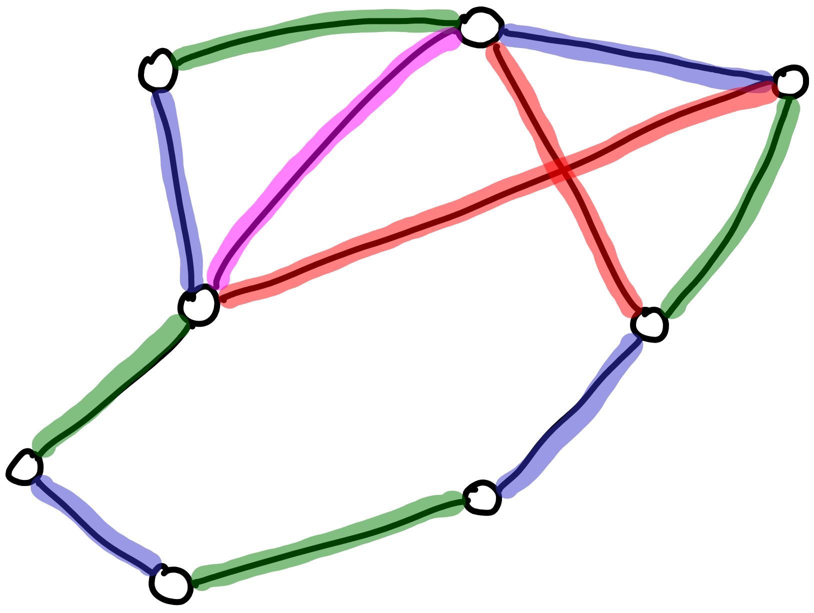
$\chi' = 4$

$\geq \Delta$



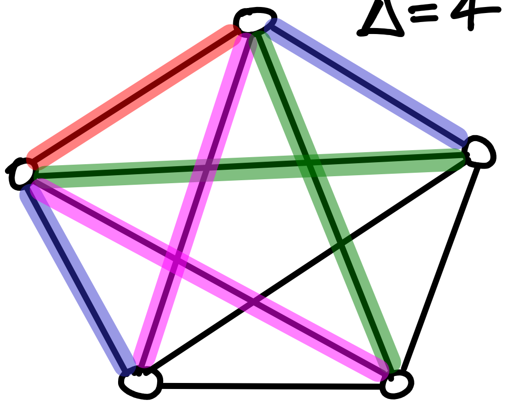
EDGE COLORING - no adjacent edges w same color

$\chi'(G)$: min # colors : edge-chromatic #



$\chi' = 4$
$\geq \Delta$

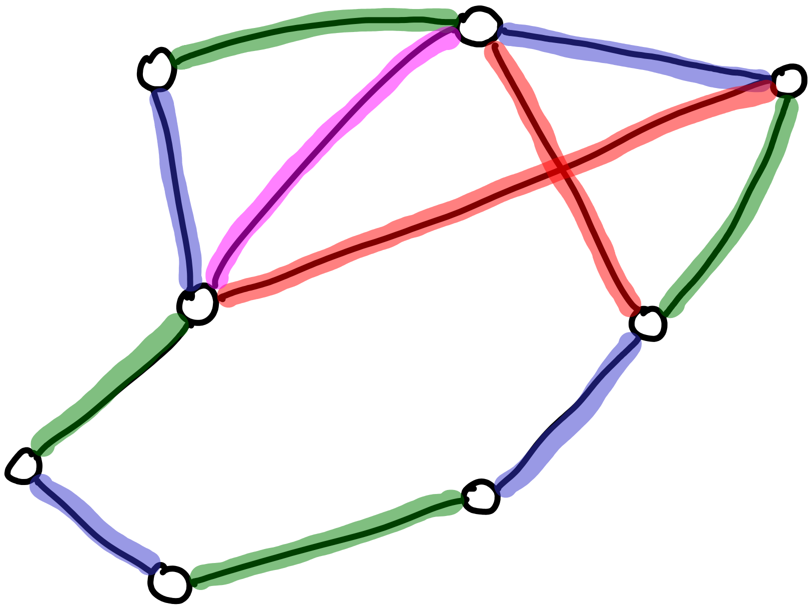
$\Delta = 4$



- Can't add more
- Can add one red & will need 2 more colors

EDGE COLORING - no adjacent edges w same color

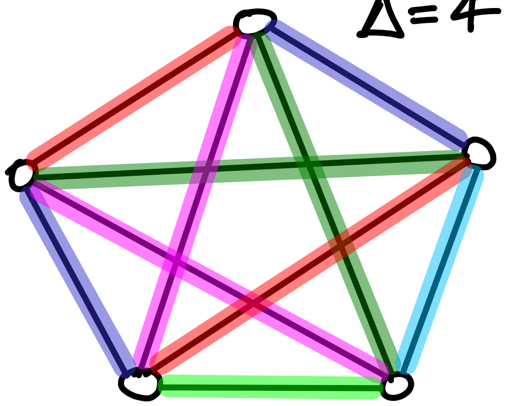
$\chi'(G)$: min # colors : edge-chromatic #



$\chi' = 4$

$\geq \Delta$

$\Delta = 4$

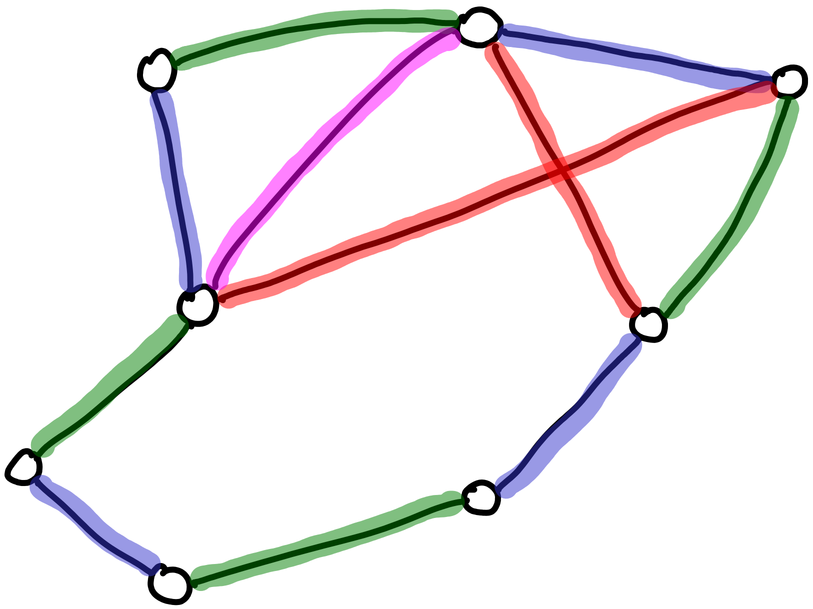


- Can't add more
- Can add one red & will need 2 more colors

$4 < \chi' \leq 6$

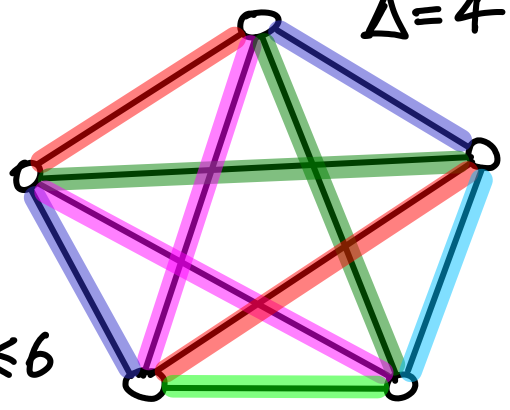
EDGE COLORING - no adjacent edges w same color

$\chi'(G)$: min # colors: edge-chromatic #



$\chi' = 4$

$\geq \Delta$



$\chi' = 5$?

Alternate proof for $\chi' > 4$

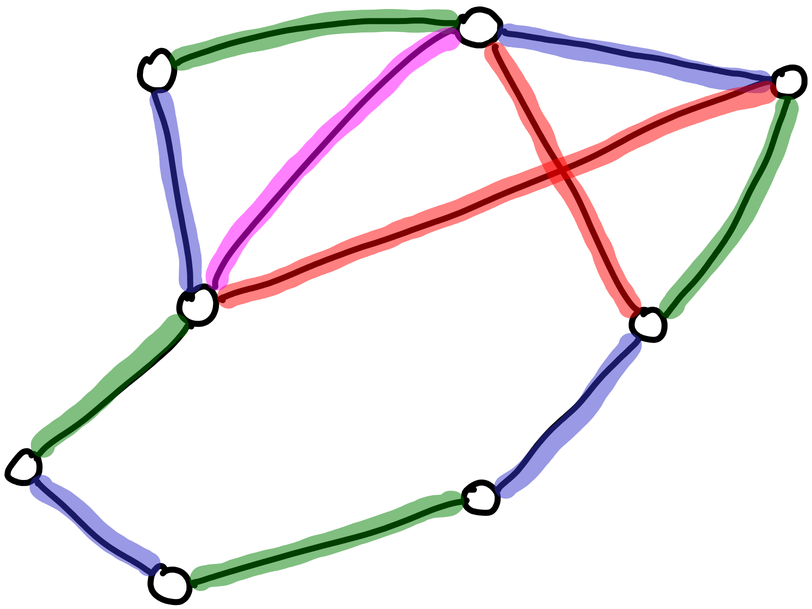
10 edges → average #edges per color = 2.5

So some color goes on 3 edges: PROBLEM

χ' can't always achieve Δ

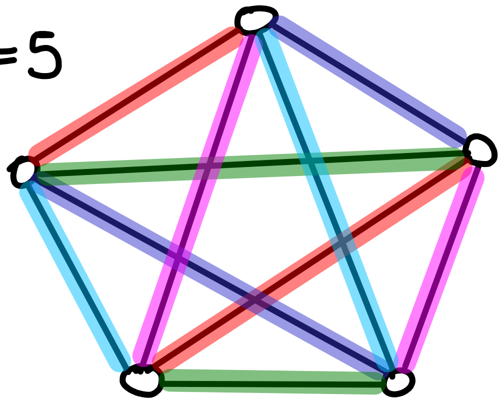
EDGE COLORING - no adjacent edges w same color

$\chi'(G)$: min # colors : edge-chromatic #



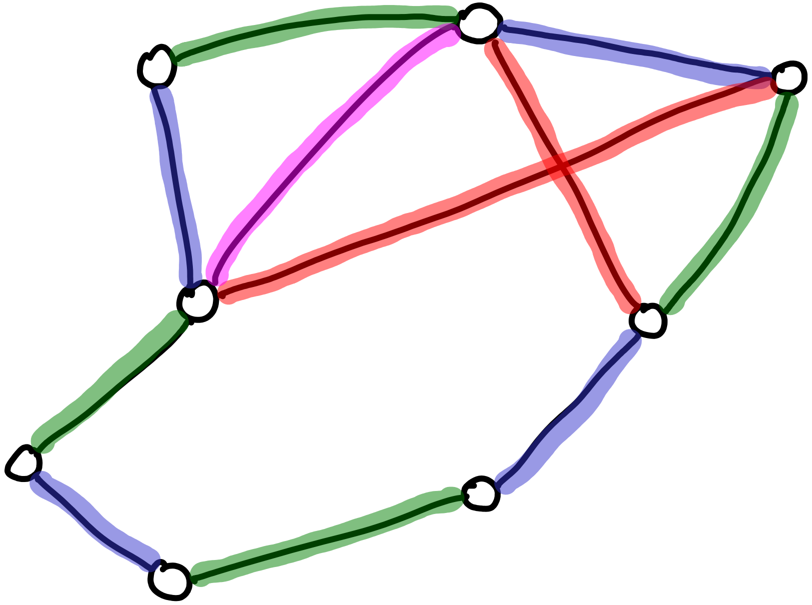
$\chi' = 4 \geq \Delta$

$\chi' = 5$

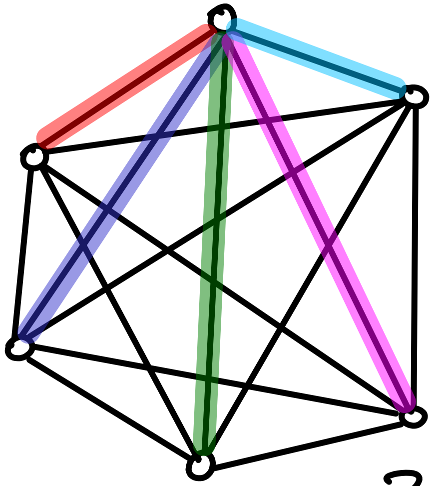


EDGE COLORING - no adjacent edges w same color

$\chi'(G)$: min # colors : edge-chromatic #

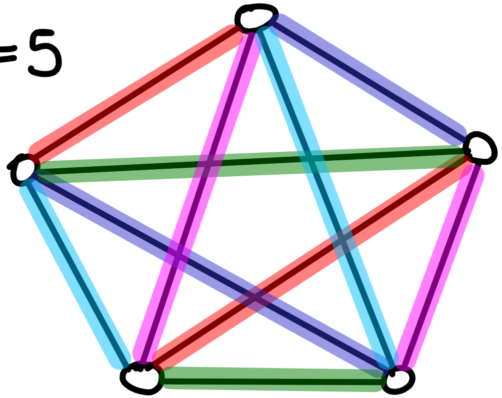


$\chi' = 4$
$\geq \Delta$



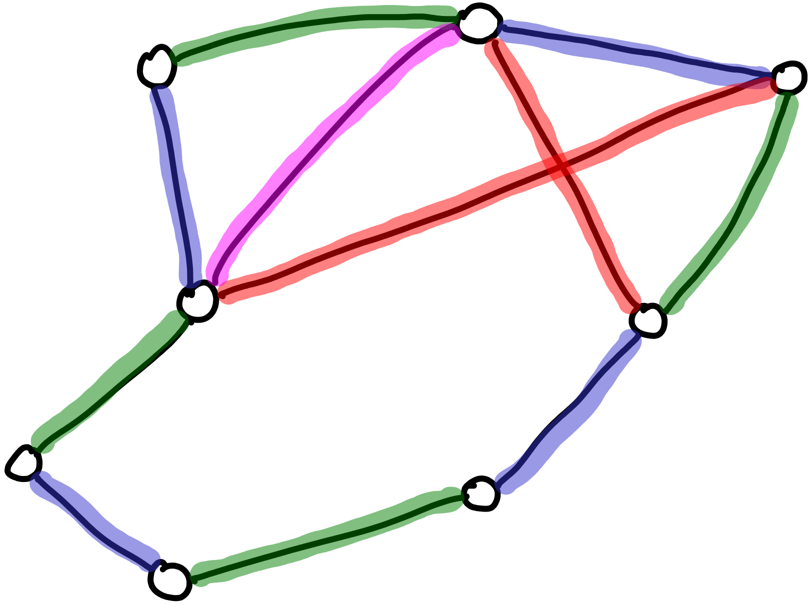
?

$\chi' = 5$



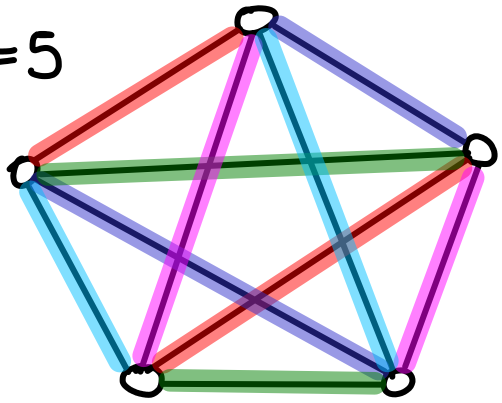
EDGE COLORING – no adjacent edges w same color

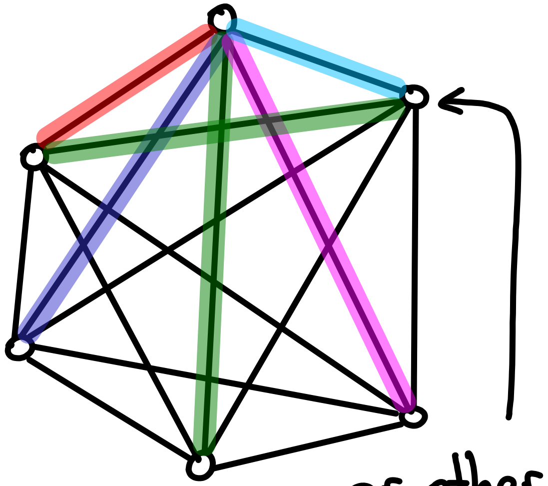
$\chi'(G)$: min # colors : edge-chromatic #



$\chi' = 4$
$\geq \Delta$

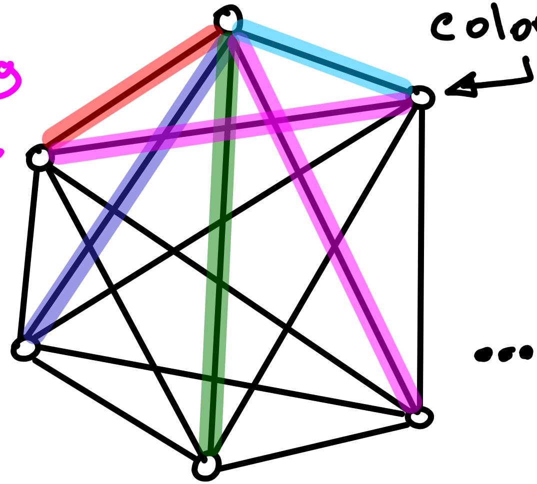
$\chi' = 5$





or other color ?

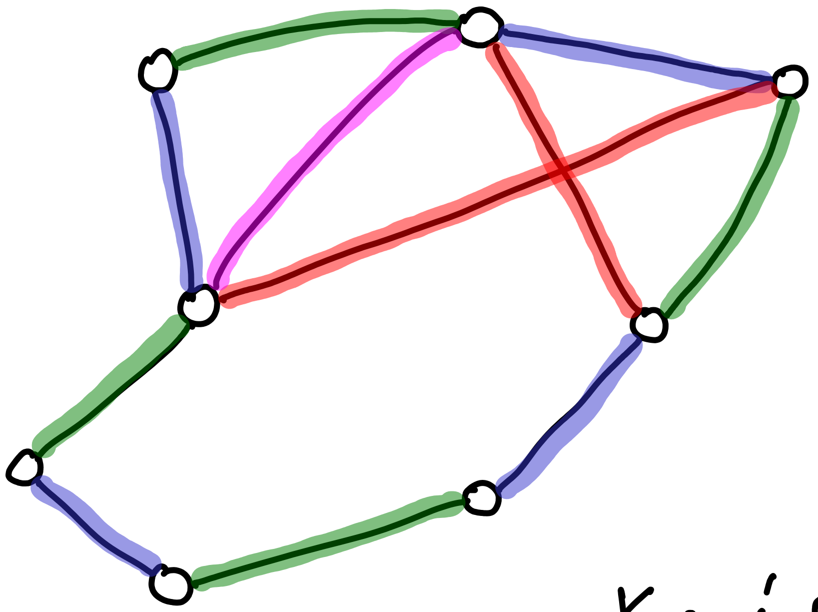
wlog

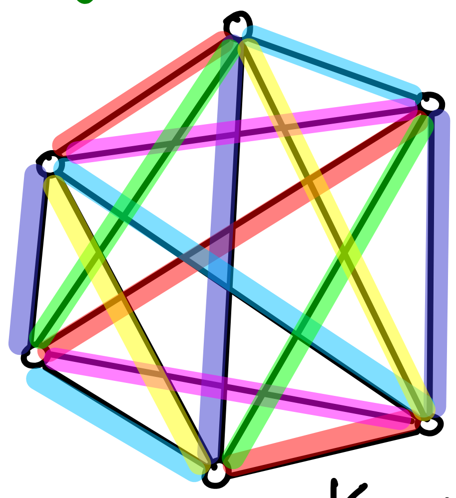


...?

EDGE COLORING - no adjacent edges w same color

$\chi'(G)$: min # colors : edge-chromatic #

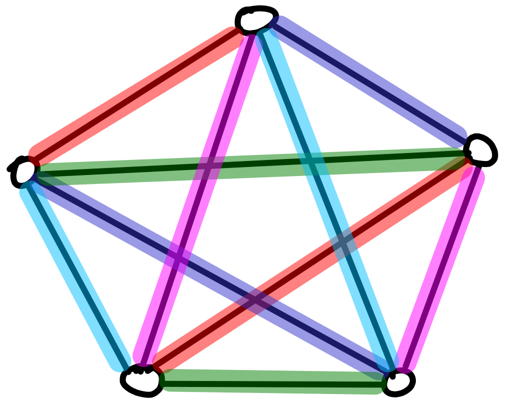




$K_6 : \chi' \leq 6$

$\chi' = 4$
$\geq \Delta$

$K_5 : \chi' = 5$



EDGE COLORING – no adjacent edges w same color

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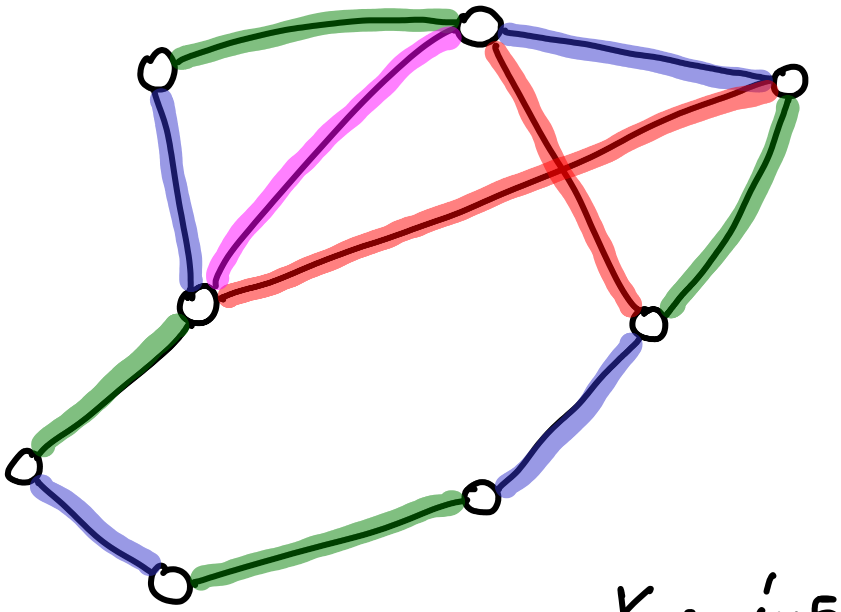
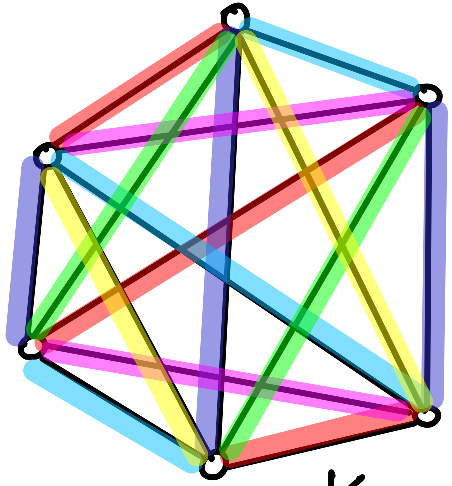


$\chi' = 4 \geq \Delta$



$K_6 : \chi' \leq 6$

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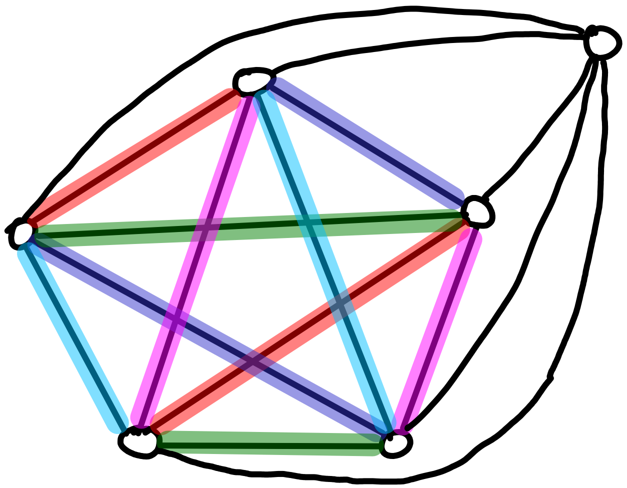
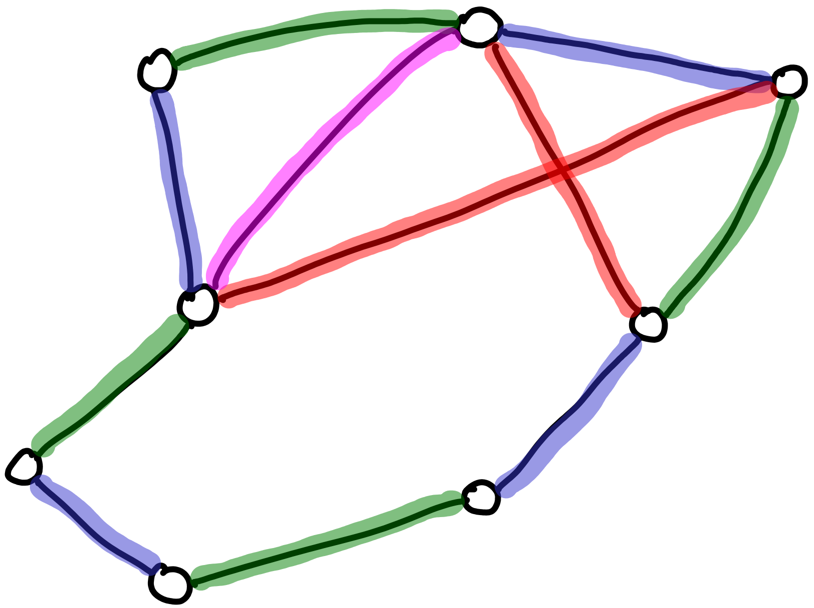


K_6

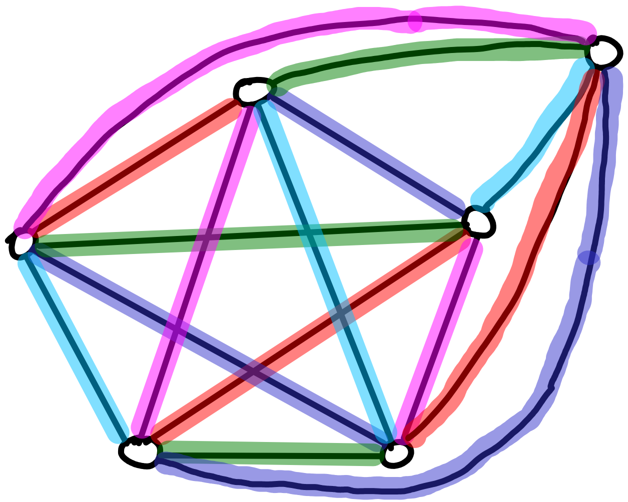
Easy trick: extend

EDGE COLORING - no adjacent edges w same color

$\chi'(G)$: min # colors : edge-chromatic #



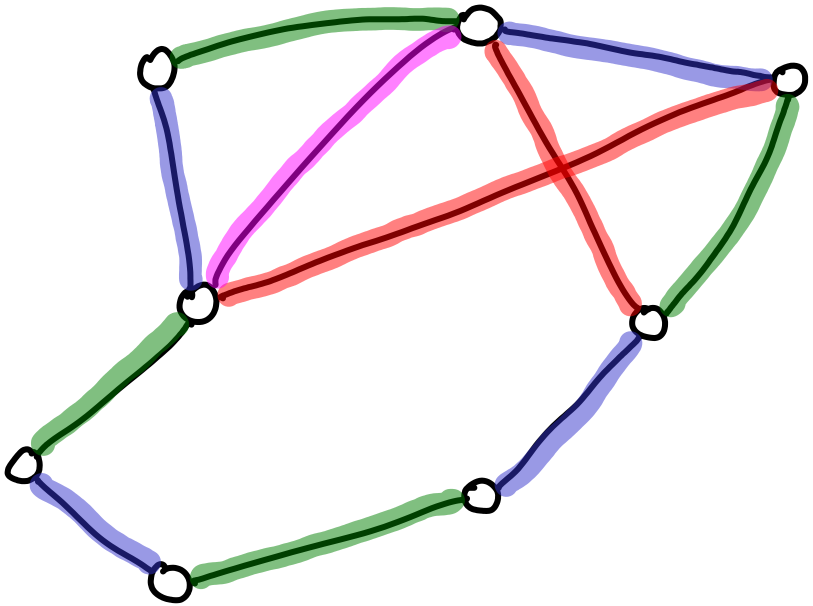
$\chi' = 4$
$\geq \Delta$



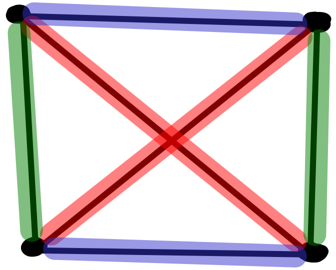
$\chi' = 5$

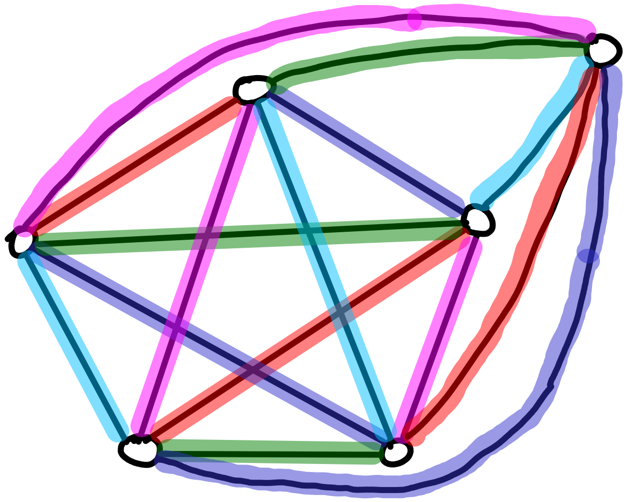
EDGE COLORING - no adjacent edges w same color

$\chi'(G)$: min # colors : edge-chromatic #



$\chi' = 4$
$\geqslant \Delta$



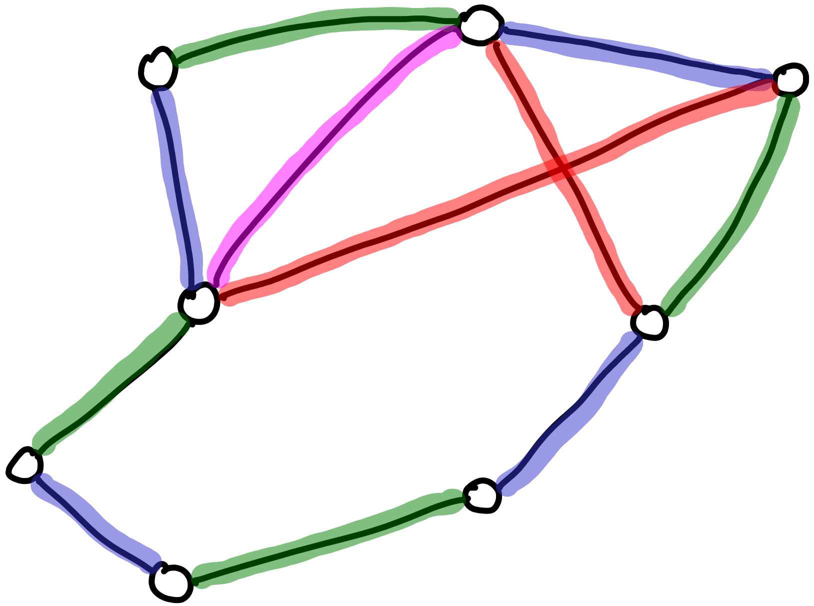


$\chi' = 5$

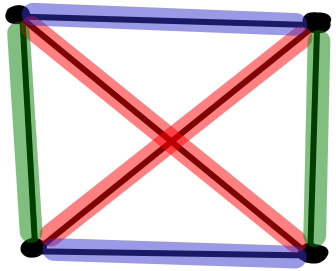
K_3	$\Delta = 2$	$\chi' = 3$
K_4	$\Delta = 3$	$\chi' = 3$
K_5	$\Delta = 4$	$\chi' = 5$
K_6	$\Delta = 5$	$\chi' = 5$!

EDGE COLORING - no adjacent edges w same color

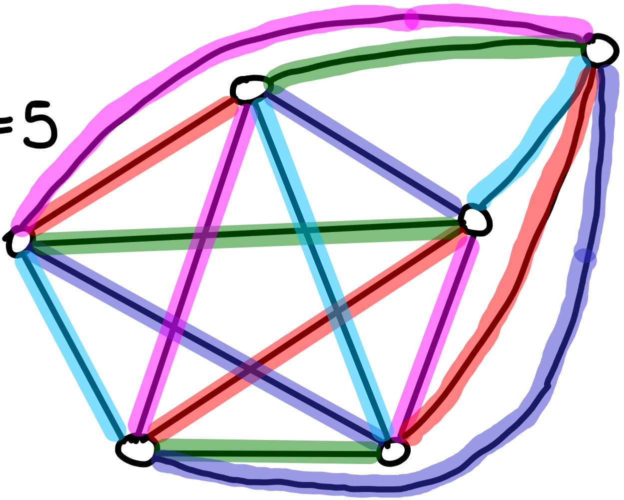
$\chi'(G)$: min # colors: edge-chromatic #



$\chi' = 4 \geq \Delta$



$\chi' = 5$



K_3 $\Delta = 2$ $\chi' = 3$

K_4 $\Delta = 3$ $\chi' = 3$

K_5 $\Delta = 4$ $\chi' = 5$

K_6 $\Delta = 5$ $\chi' = 5$!

K_7 can't continue extension

K_n : $f(n)$ and/or $f(\Delta)$?

Vizing's Thm: for all graphs, $\chi' \leq \Delta + 1$ (close to trivial lower bound)

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There are proofs by induction on E, V and Δ

most common

Equivalent to algorithm that incrementally extends G.

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Observation : If we are willing to use $\Delta+1$ colors, then every vertex has ≥ 1 "missing" color. $d(v) \le \Delta < \Delta+1$

no incident edge has that color

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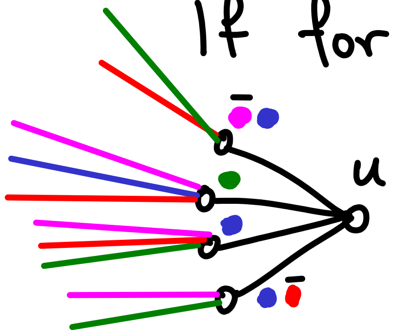
implies

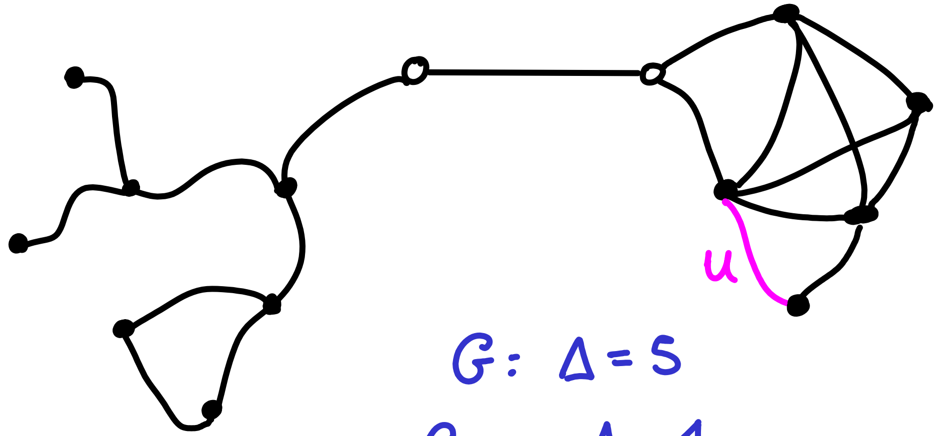


If for any $G' = G - u$, each neighbor of u in G can contribute one of its missing colors to a list of size $\geq d(u)$ then DONE.

list(u):

(induction on V)

Observations for induction on E.



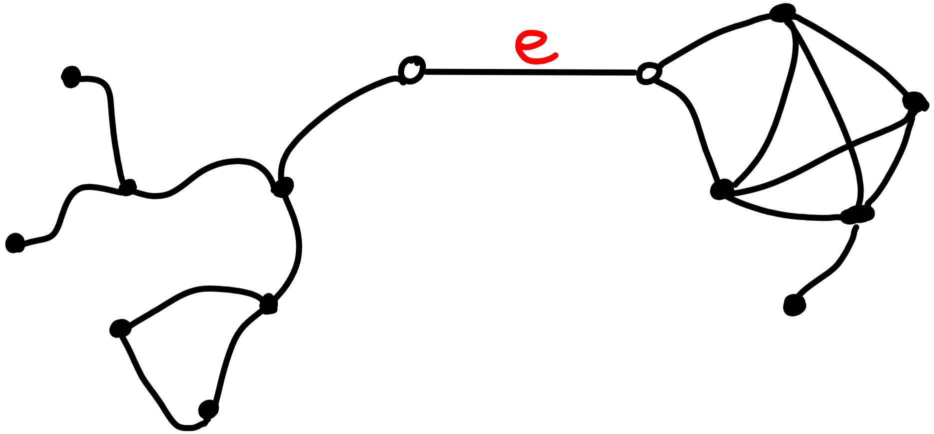
$G: \Delta = 5$

$G-u: \Delta = 4$

- If $G-u$ has $\Delta - 1$, it uses $\leq \Delta$ colors. Pick a new color for u.

  – DONE

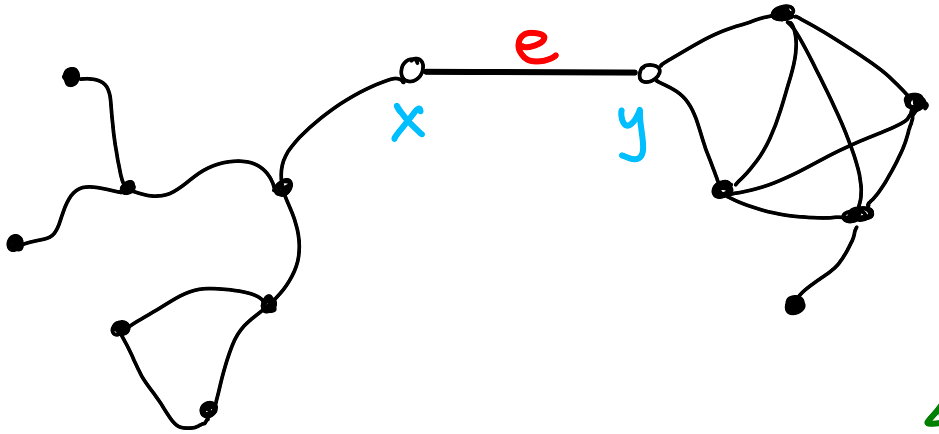
Observations for induction on E.



- If $G-u$ has $\Delta-1$, it uses $\leq \Delta$ colors. Pick a new color for u.
  —DONE

- If $G-e$ is disconnected, then remove e & color the <u>2</u> components by induction.

Observations for induction on E.



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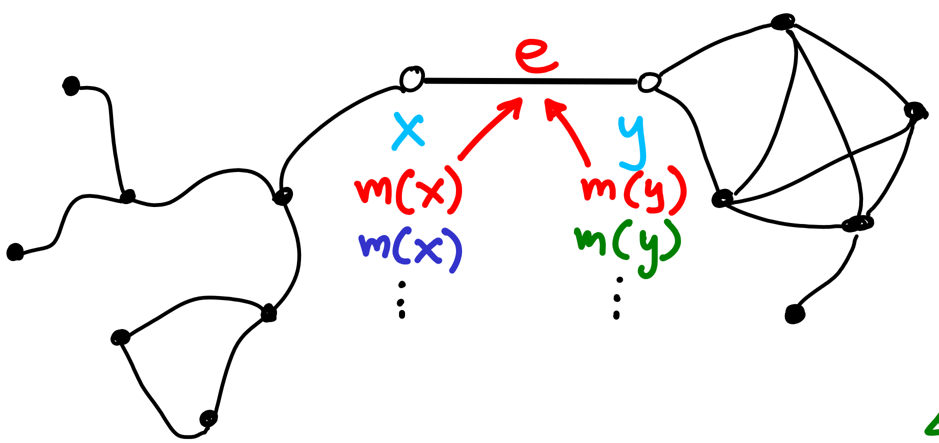
Even if components have Δ, using $\Delta+1$ colors,
x & y have degree $< \Delta$: have ≥ 2 missing colors

Observations for induction on E.



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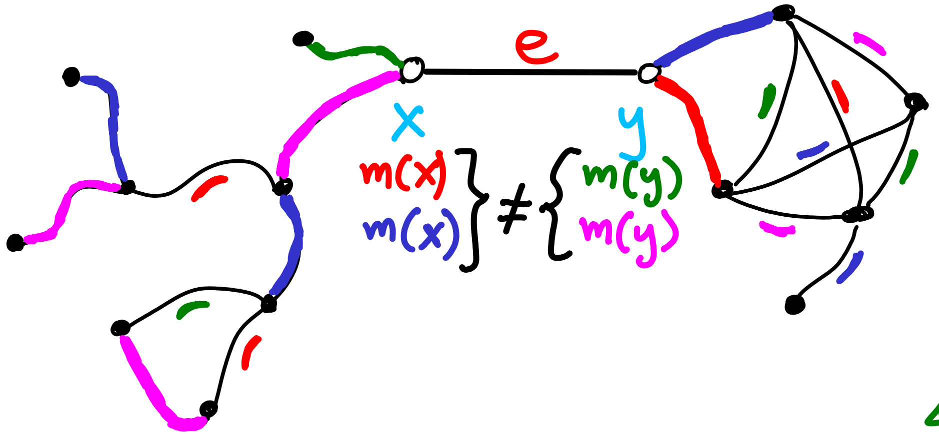
- If $G-e$ is disconnected, then remove e & color the $\underline{\underline{2}}$ components by induction.

Even if components have Δ, using $\Delta+1$ colors,

x & y have degree $< \Delta$: have ≥ 2 missing colors

If $\exists$ missing color $m(x) = m(y)$, use this for e.

Observations for induction on E.



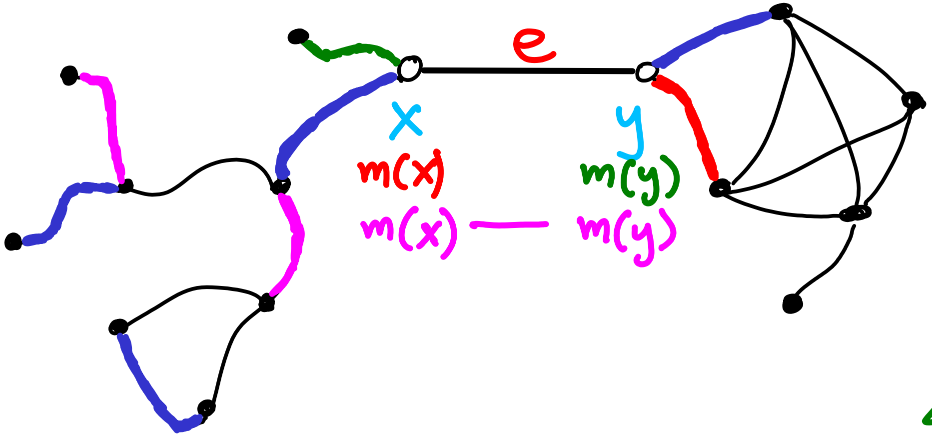
- If $G-u$ has $\Delta-1$, it uses $\leq \Delta$ colors. Pick a new color for u.
  -DONE

- If $G-e$ is disconnected, then remove e & color the 2 components by induction.

Even if components have Δ, using $\Delta+1$ colors,
x & y have degree $< \Delta$: have ≥ 2 missing colors

$\begin{cases} \text{If } \exists \text{ missing color } m(x) = m(y), \text{ use this for } e. \\ \text{Else} \ldots \end{cases}$

Observations for induction on E.



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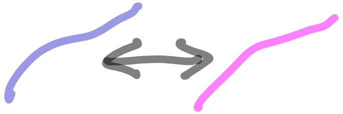
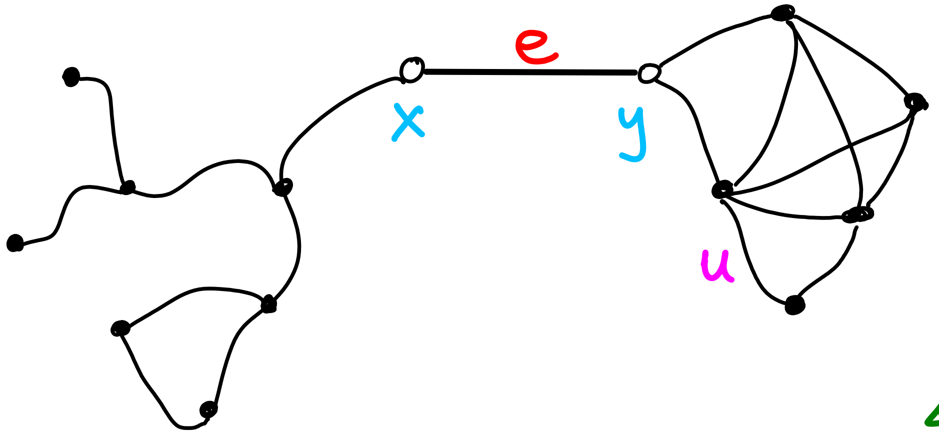
- If $\exists$ missing color $m(x) = m(y)$, use this for e.
- Else, remap colors in one component s.t $m(x) = m(y)$
  (permute)



Observations for induction on E.



- If $G-u$ has $\Delta - 1$, it uses $\leq \Delta$ colors. Pick a new color for u.
  - DONE

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$\left\{ \begin{array}{l} \text{If } \exists \text{ missing color } m(x) = m(y), \text{ use this for } e. \\ \text{Else, remap colors in one component s.t } m(x) = m(y) \end{array} \right.$

Conclusion: Induction on edges: trivial if any edge not on a cycle, or uniquely increases Δ

Proof of Vizing's thm by induction on edges

(by constructive algorithm)

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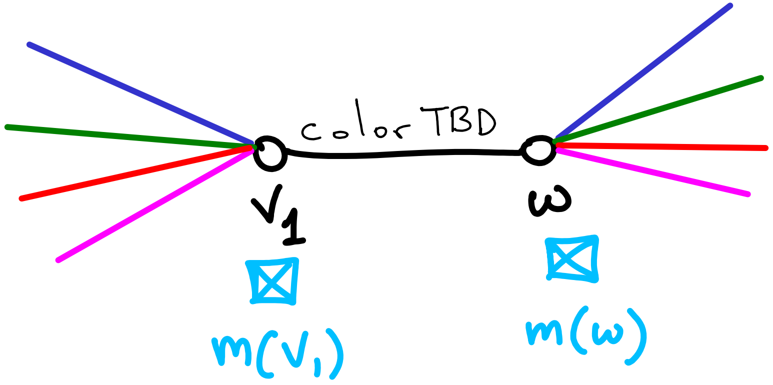
(by constructive algorithm)

Assume we have colored $G - \{\overline{v_1 w}\}$

We already know if $\exists\ m(v_1) = m(w)$ then DONE.



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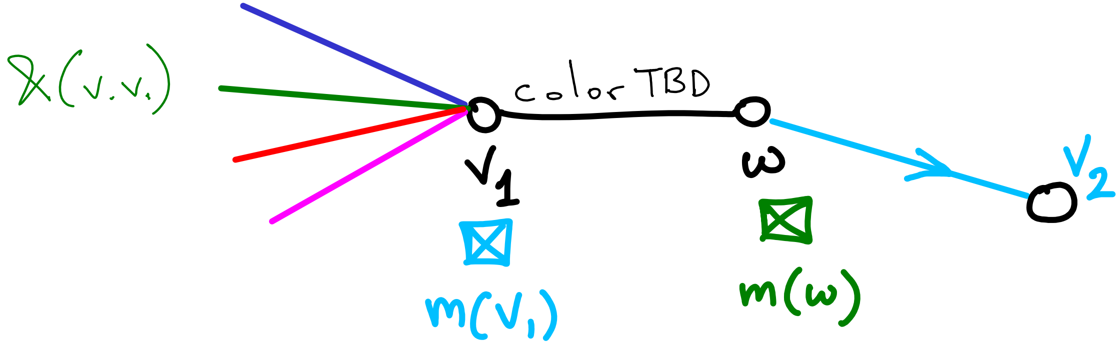
So pick any $m(v_1)$ ... w must have a neighbor v_2 with color $m(v_1)$



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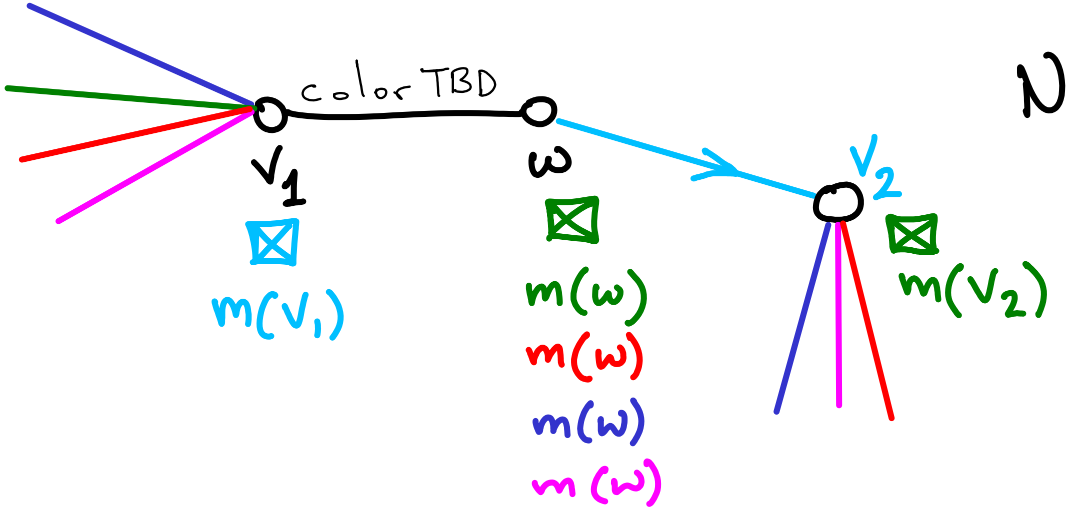


Now see if $\exists$ $m(v_2) = m(w)$

↳ If yes, we're DONE...

$\Downarrow$

recolor $v_2 w \rightarrow v_2 w$

color $v_1 w \rightarrow v_1 w$

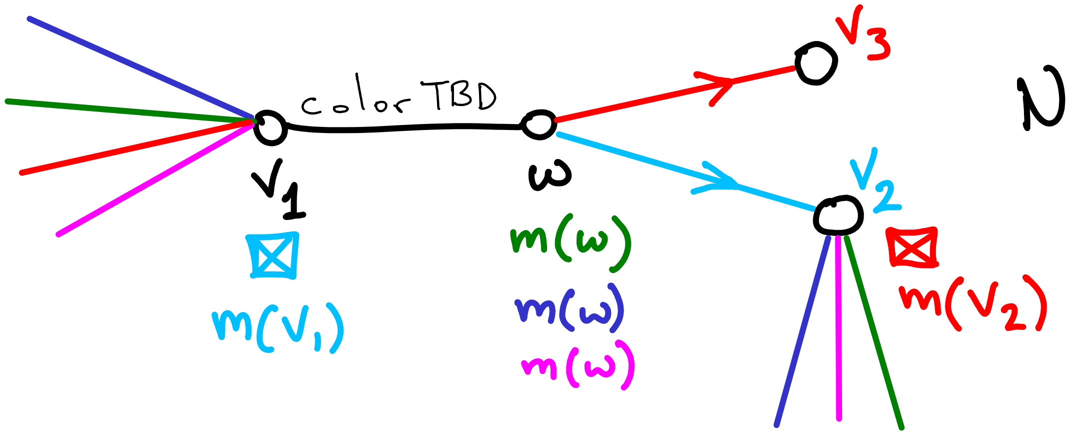
Proof of Vizing's thm by induction on edges

(by constructive algorithm)

Assume we have colored $G - \{\overline{v_1 w}\}$

We already know if $\exists\ m(v_1) = m(w)$ then DONE.

So pick any $m(v_1)$ ... w must have a neighbor v_2 with color $m(v_1)$

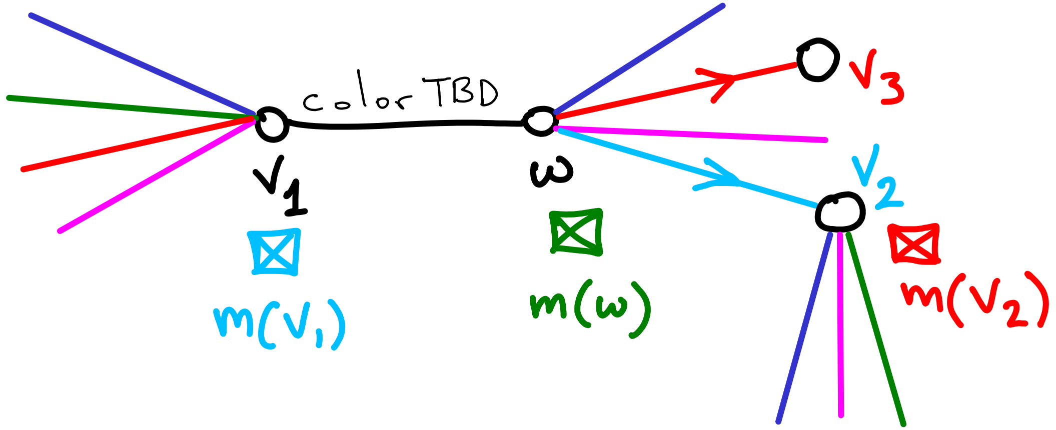


Now see if $\exists\ m(v_2) = m(w)$

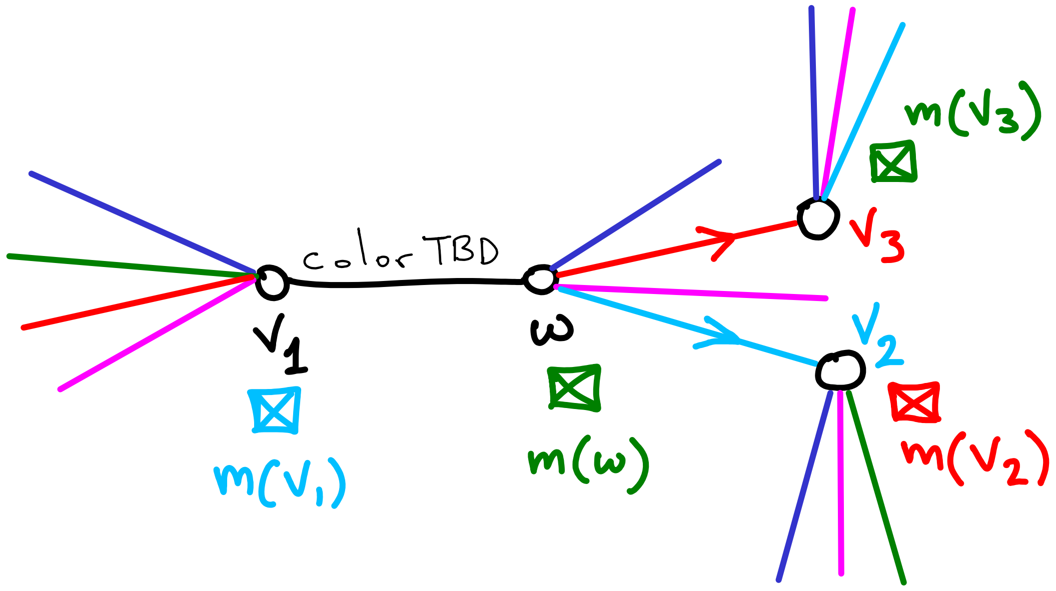
↳ If yes, we're DONE, else

$\exists\ m(v_2) \neq m(w) \neq m(v_1)$

w has a neighbor v_3 ...

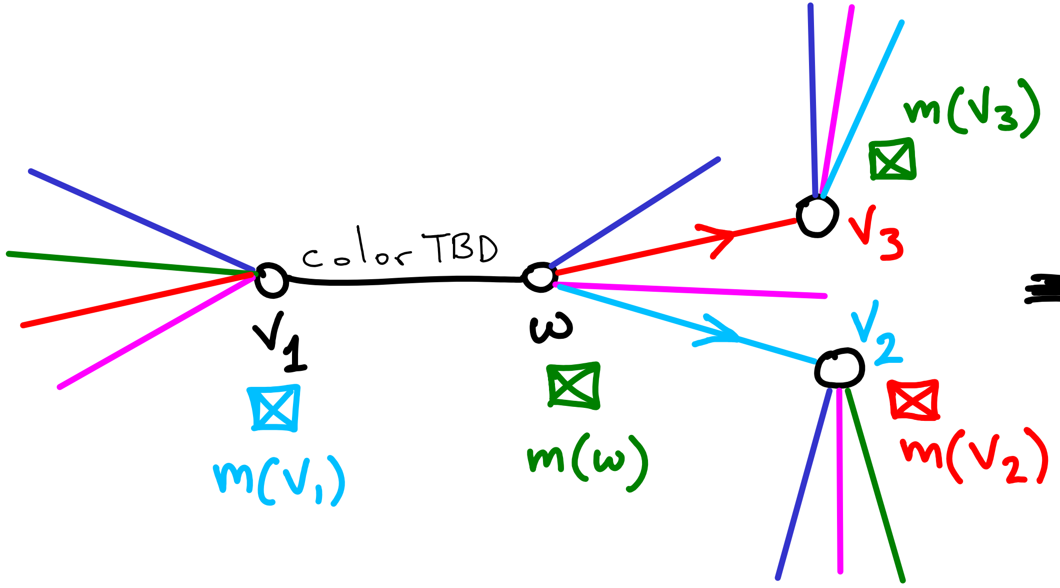


Check if $\exists\ m(v_3) = m(w)$



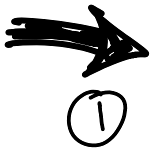
Check if $\exists$ $m(v_3) = m(w)$

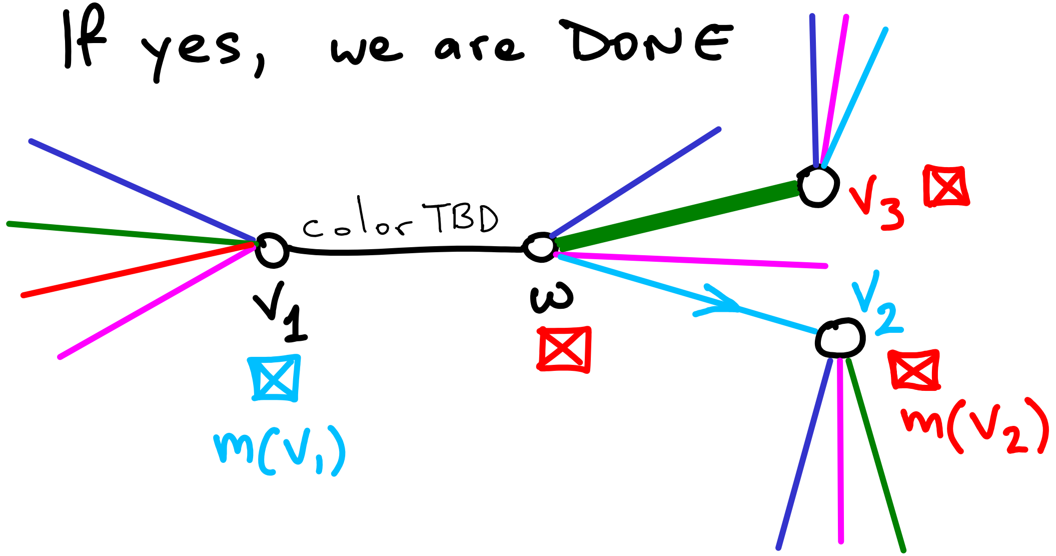
If yes, we are DONE ...



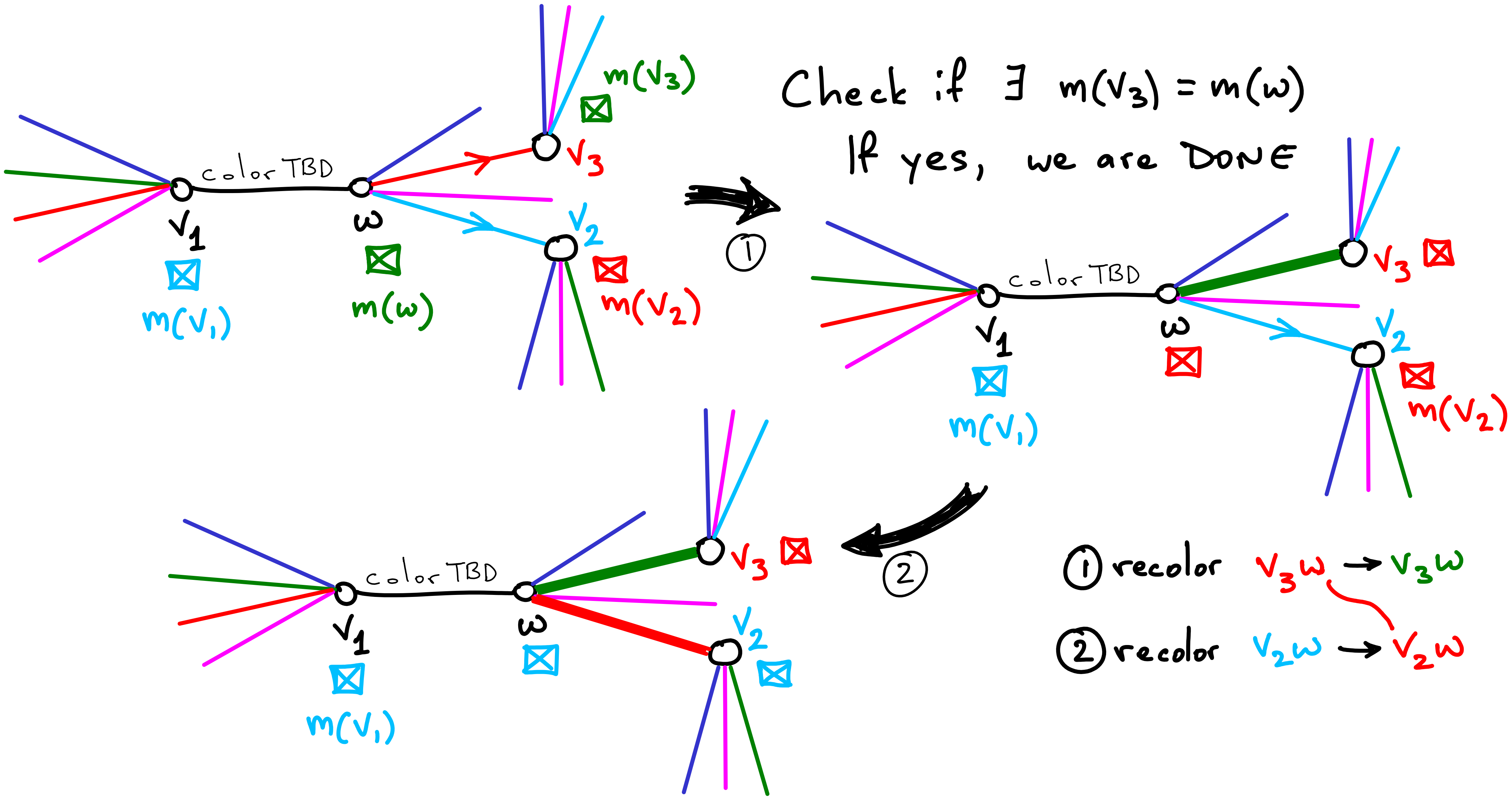
Check if $\exists$ $m(v_3) = m(w)$

If yes, we are DONE

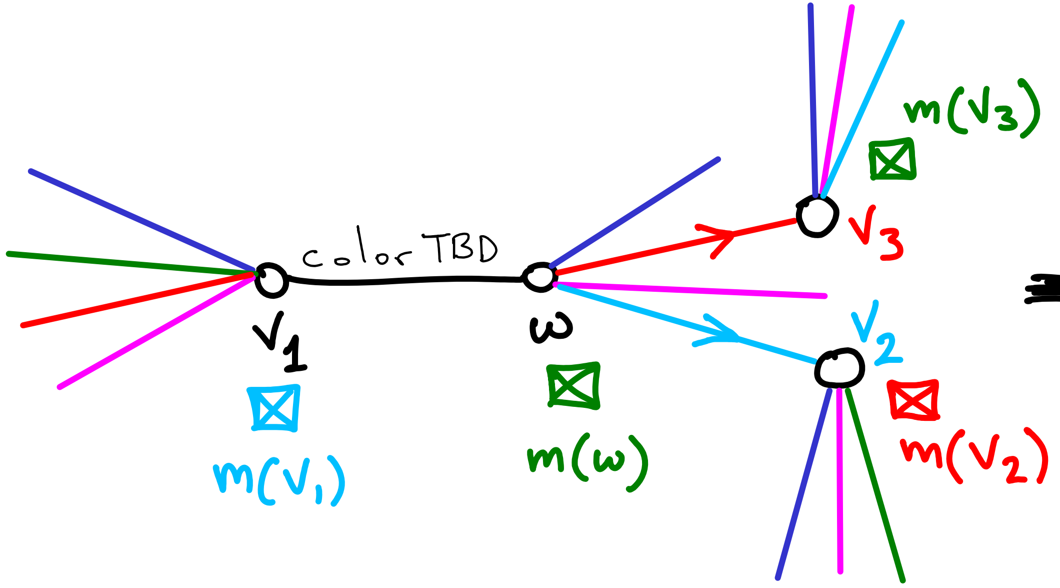




① recolor $v_3w \to v_3w$



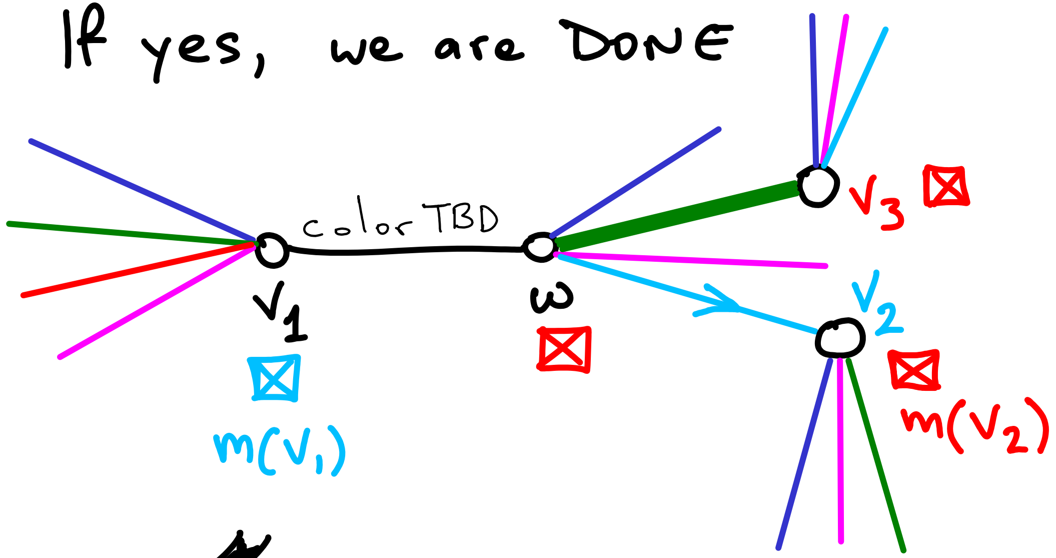
color TBD
v_1
$m(v_1)$
w
$m(w)$
$m(v_3)$
v_3
v_2
$m(v_2)$
1
Check if $\exists\ m(v_3) = m(w)$
If yes, we are DONE
2
1 recolor $v_3w \to v_3w$
2 recolor $v_2w \to v_2w$



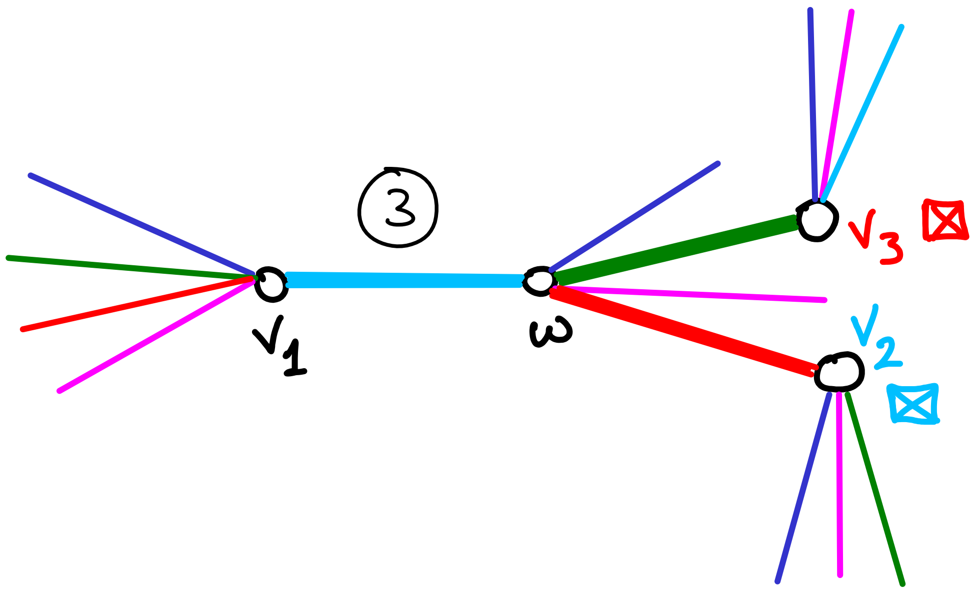
①

Check if $\exists\; m(v_3) = m(w)$

If yes, we are DONE



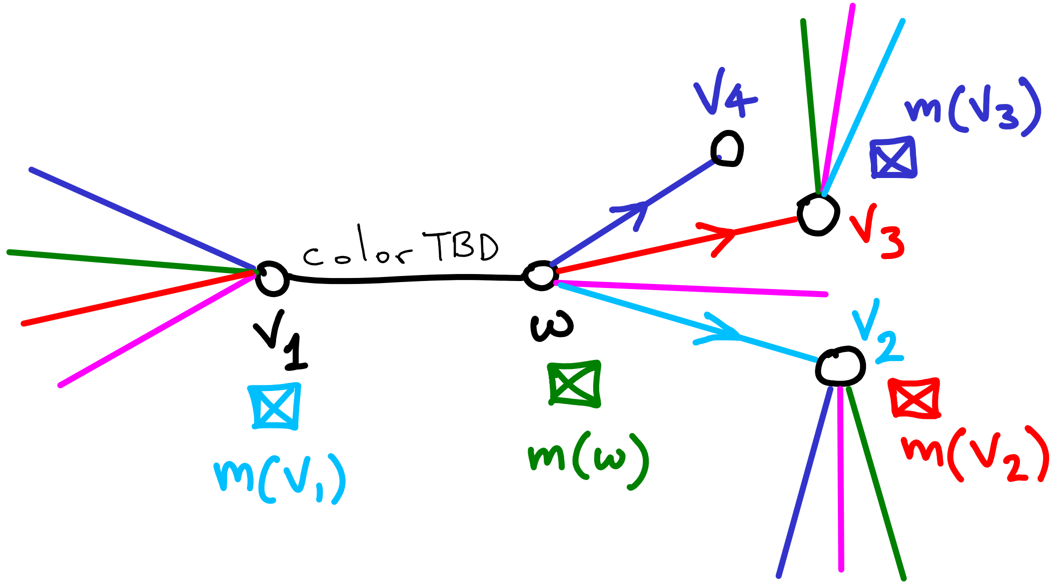
②



① recolor $v_3w \to v_3w$

② recolor $v_2w \to v_2w$

③ color $v_1w \to v_1w$



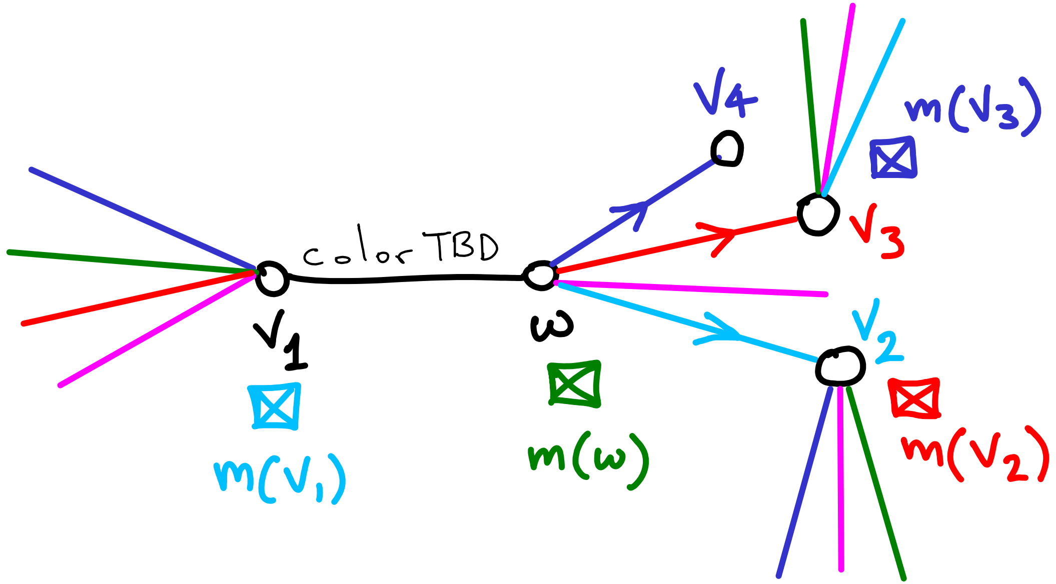
Check if $\exists$ $m(v_3) = m(w)$

If yes, we are DONE else

$\exists$ v_4 s.t edge wv_4 has color $m(v_3)$

& we keep going

(but not forever)



Check if $\exists\ m(v_3) = m(w)$

If yes, we are DONE else

$\exists\ v_4$ s.t edge wv_4 has

color $m(v_3)$

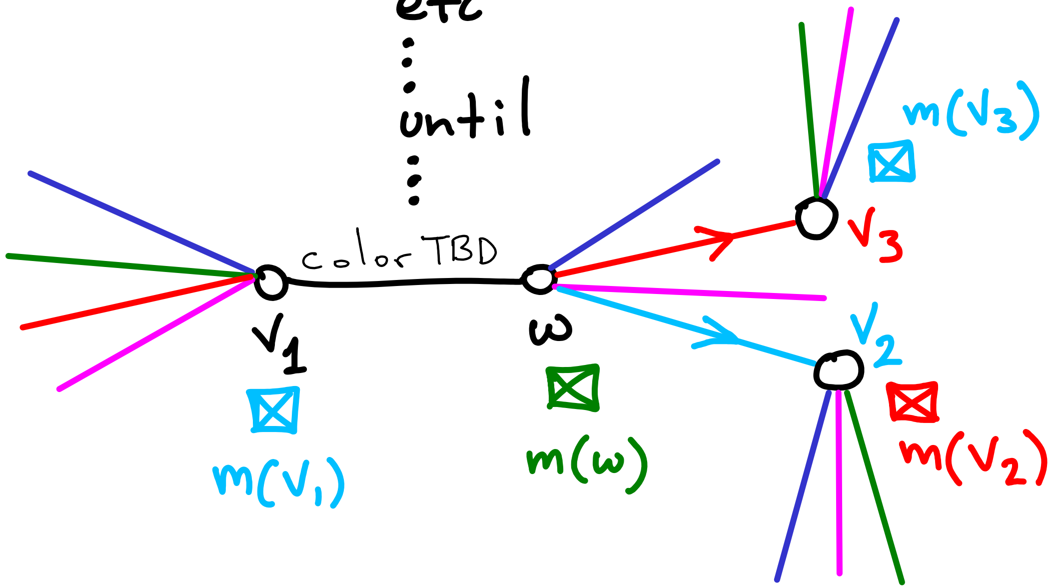
⋮

etc

⋮

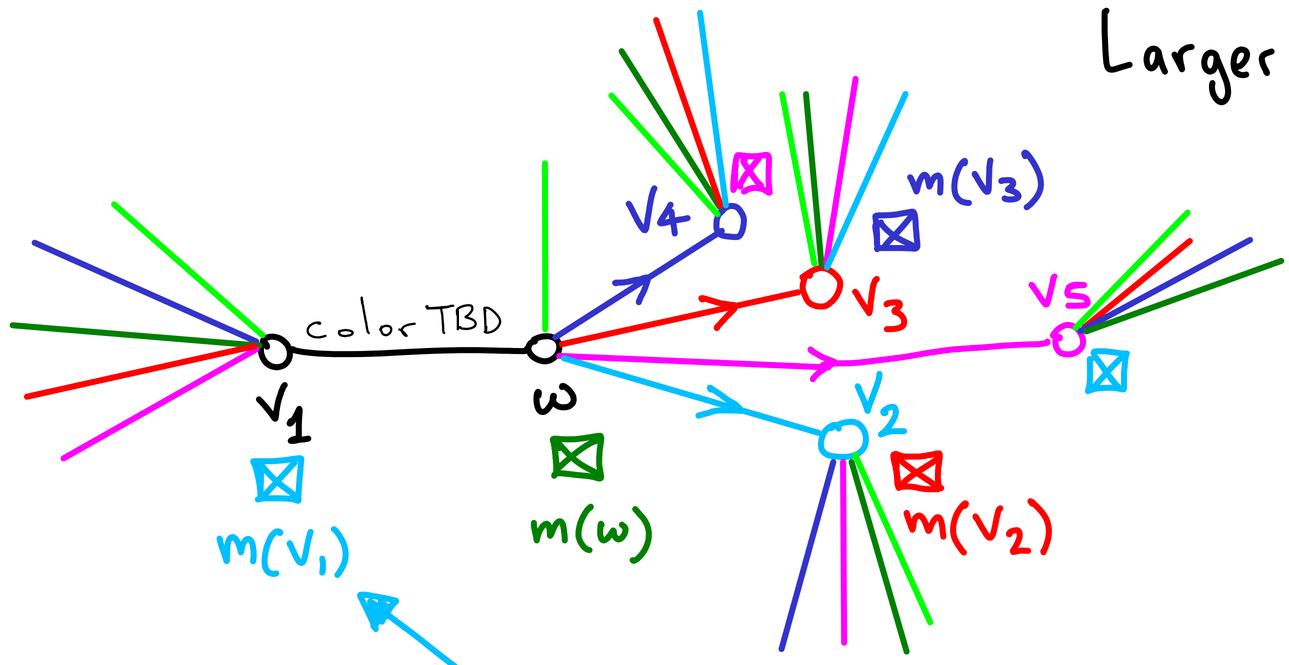
until

⋮



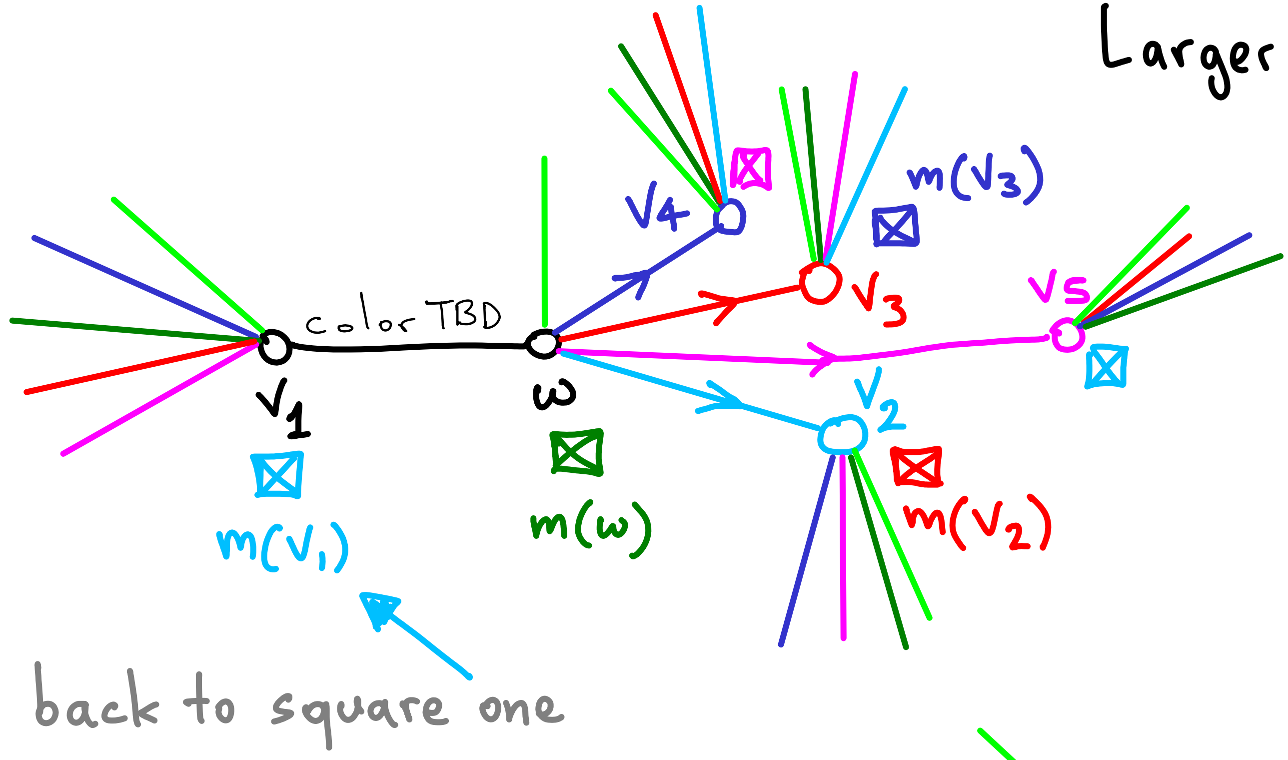
We're back to a missing color that we saw already

Larger example of inevitable cycle

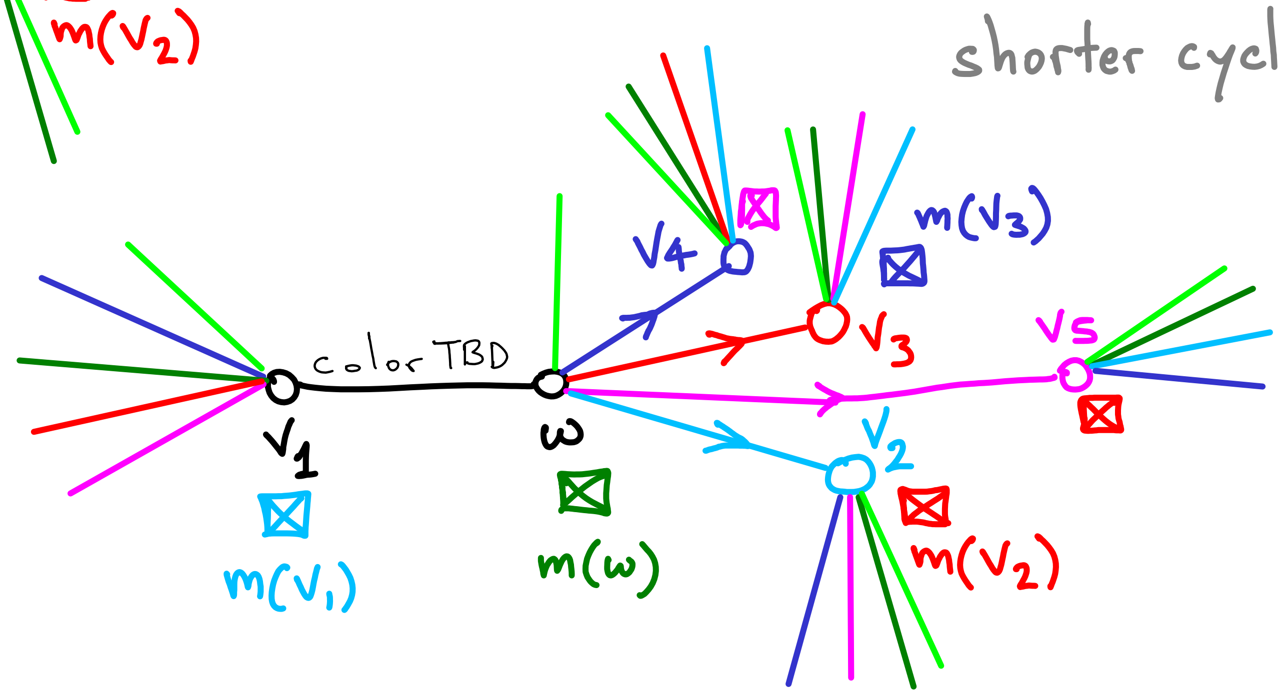


back to square one

Larger examples of inevitable cycle

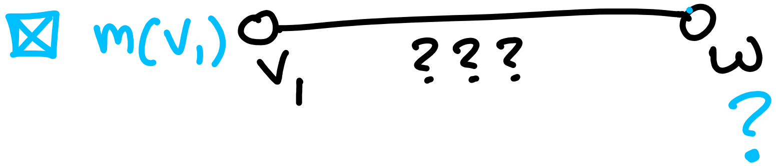


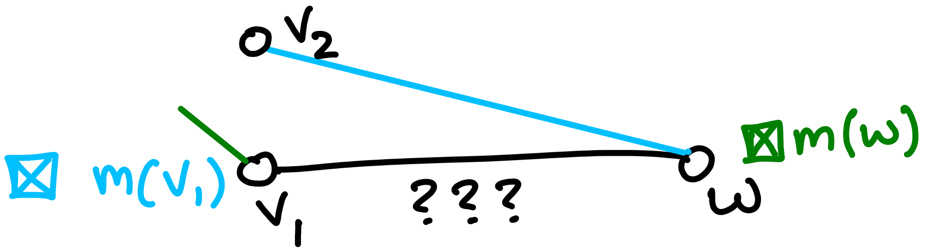
shorter cycle



Given: $\boxtimes\, m(v_1)$

If $\boxtimes\, m(w)$ DONE

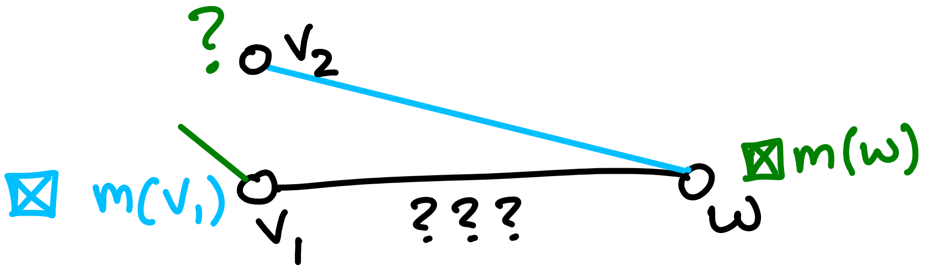




Given: $\boxtimes\, m(v_1)$

If $\boxtimes\, m(w)$ DONE

else $\exists\, wv_2$ & $\boxtimes\, m(w)$

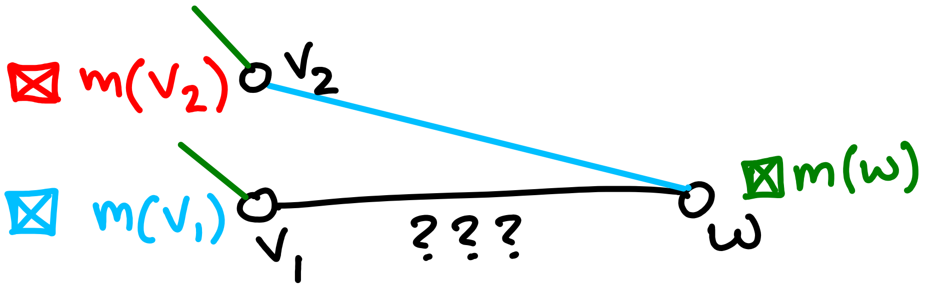


Given: $\boxtimes\ m(v_1)$

If $\boxtimes m(w)$ DONE

else $\exists\ wv_2$ & $\boxtimes m(w)$

If $\boxtimes m(v_2)$ DONE



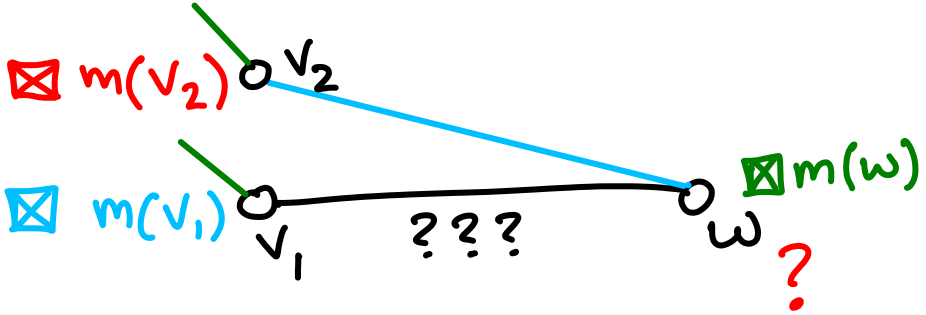
Given: $\boxtimes\, m(v_1)$

If $\boxtimes\, m(w)$ DONE

else $\exists\, wv_2$ & $\boxtimes\, m(w)$

If $\boxtimes\, m(v_2)$ DONE

else $\exists\, \overline{v_2 v_?}$ & $\boxtimes\, m(v_2)$



Given: $\boxtimes\, m(v_1)$

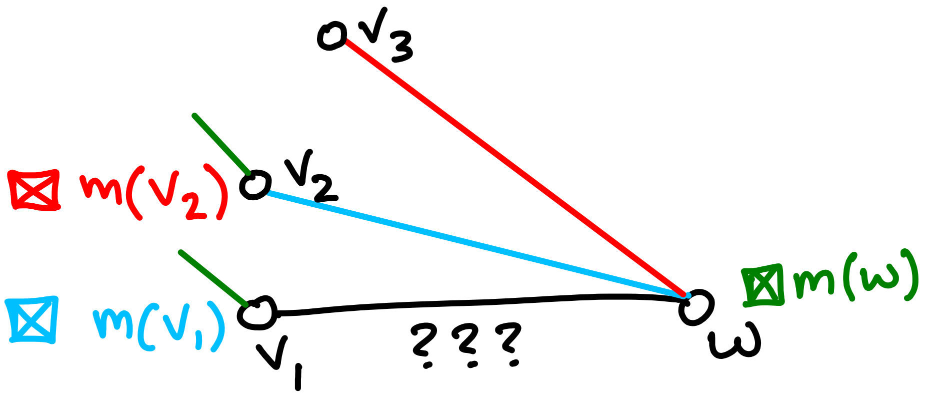
If $\boxtimes\, m(w)$ DONE

else $\exists\, wv_2$ & $\boxtimes\, m(w)$

If $\boxtimes\, m(v_2)$ DONE

else $\exists\, \overline{v_2 v_?}$ & $\boxtimes\, m(v_2)$

If $\boxtimes\, m(w)$ DONE



Given: $\boxtimes\, m(v_1)$

If $\boxtimes\, m(w)$ DONE

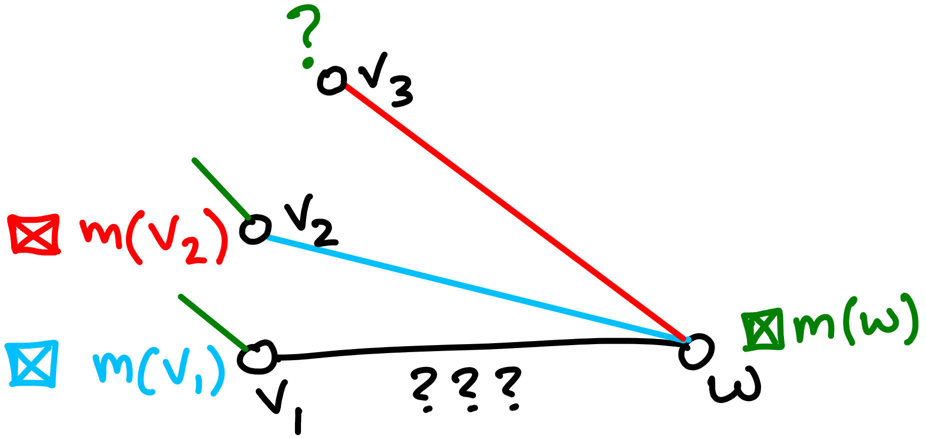
else $\exists\, wv_2$ & $\boxtimes\, m(w)$

If $\boxtimes\, m(v_2)$ DONE

else $\exists\, \overline{v_2 v_?}$ & $\boxtimes\, m(v_2)$

If $\boxtimes\, m(w)$ DONE

else $\exists\, wv_3$



Given: $\boxtimes\ m(v_1)$

If $\boxtimes m(w)$ DONE

else $\exists\ wv_2$ & $\boxtimes m(w)$

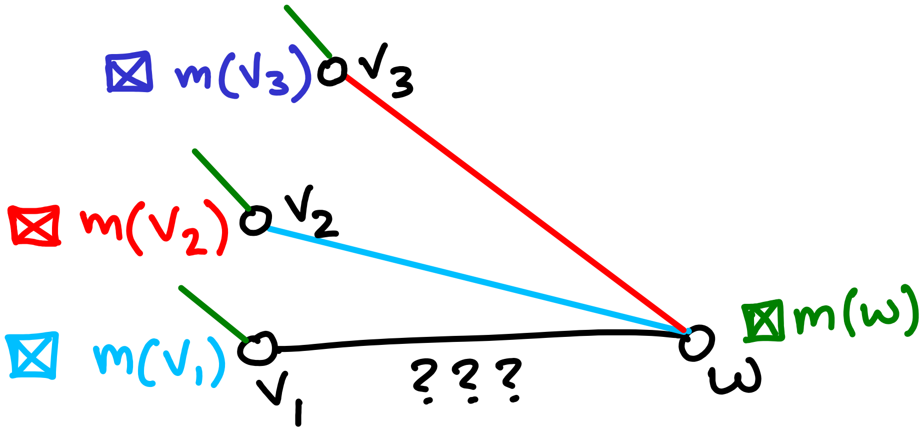
If $\boxtimes m(v_2)$ DONE

else $\exists\ \overline{v_2 v_?}$ & $\boxtimes m(v_2)$

If $\boxtimes m(w)$ DONE

else $\exists\ wv_3$

If $\boxtimes m(v_3)$ DONE



Given: $\boxtimes\ m(v_1)$

If $\boxtimes m(w)$ DONE

else $\exists\ wv_2$ & $\boxtimes m(w)$

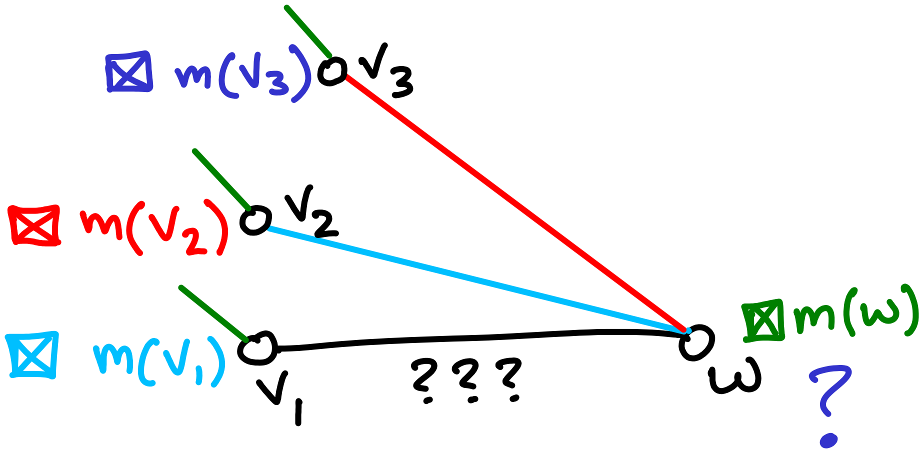
If $\boxtimes m(v_2)$ DONE

else $\exists\ \overline{v_2 v_?}$ & $\boxtimes m(v_2)$

If $\boxtimes m(w)$ DONE

else $\exists\ wv_3$

If $\boxtimes m(v_3)$ DONE $\Rightarrow \exists \boxtimes m(v_3)$



Given: $\boxtimes\ m(v_1)$

If $\boxtimes m(w)$ DONE

else $\exists\ wv_2$ & $\boxtimes m(w)$

If $\boxtimes m(v_2)$ DONE

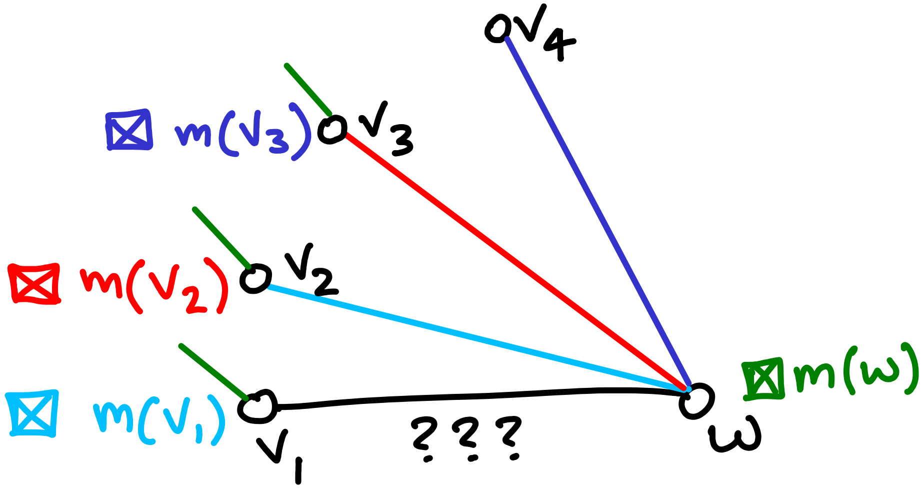
else $\exists\ \overline{v_2 v_?}$ & $\boxtimes m(v_2)$

If $\boxtimes m(w)$ DONE

else $\exists\ wv_3$

If $\boxtimes m(v_3)$ DONE $\Rightarrow \exists \boxtimes m(v_3)$

If $\boxtimes m(w)$ DONE



Given: $\boxtimes\, m(v_1)$

If $\boxtimes m(w)$ DONE

else $\exists\, wv_2$ & $\boxtimes m(w)$

    If $\boxtimes m(v_2)$ DONE

    else $\exists\, \overline{v_2 v_?}$ & $\boxtimes m(v_2)$

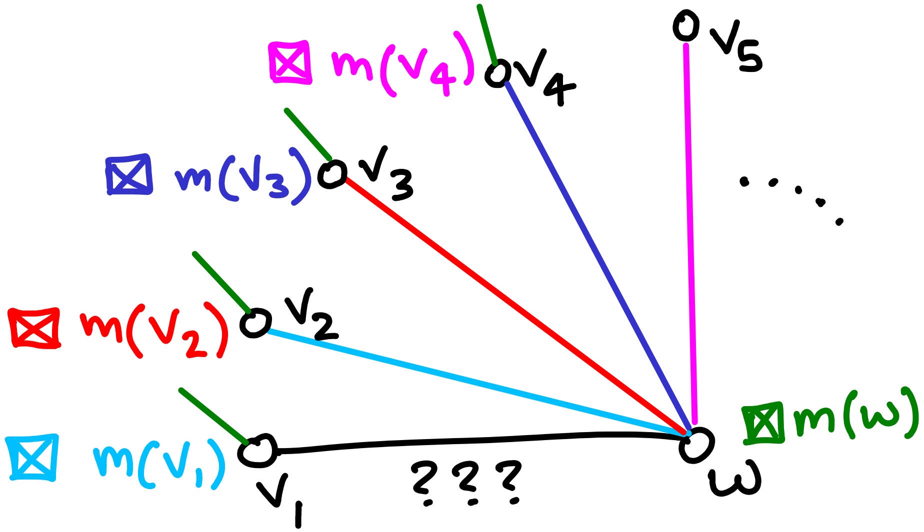
        If $\boxtimes m(w)$ DONE

        else $\exists\, wv_3$

            If $\boxtimes m(v_3)$ DONE $\Rightarrow \exists \boxtimes m(v_3)$

            If $\boxtimes m(w)$ DONE

            else $\exists\, wv_4$



Given: $\boxtimes\ m(v_1)$

If $\boxtimes m(w)$ DONE

else $\exists\ wv_2$ & $\boxtimes m(w)$

If $\boxtimes m(v_2)$ DONE

else $\exists\ \overline{v_2 v_?}$ & $\boxtimes m(v_2)$

If $\boxtimes m(w)$ DONE

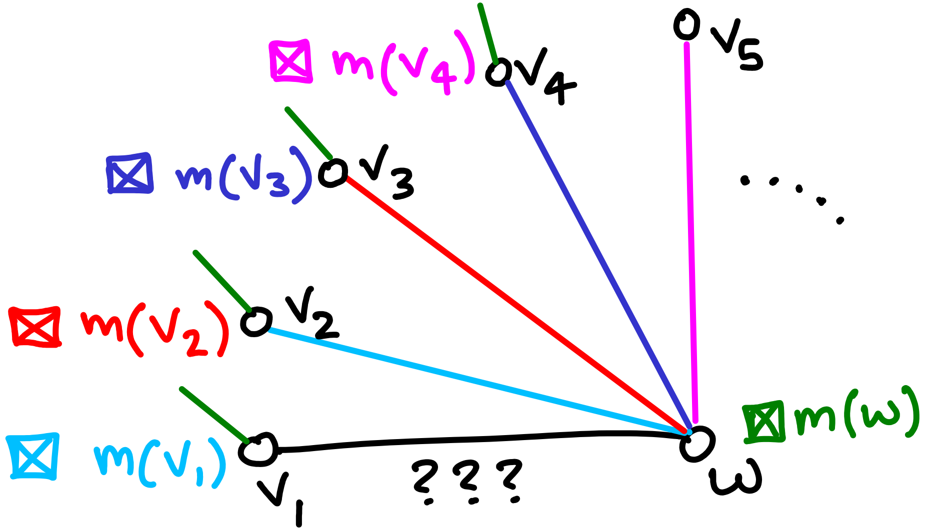
else $\exists\ wv_3$

If $\boxtimes m(v_3)$ DONE $\Rightarrow \exists \boxtimes m(v_3)$

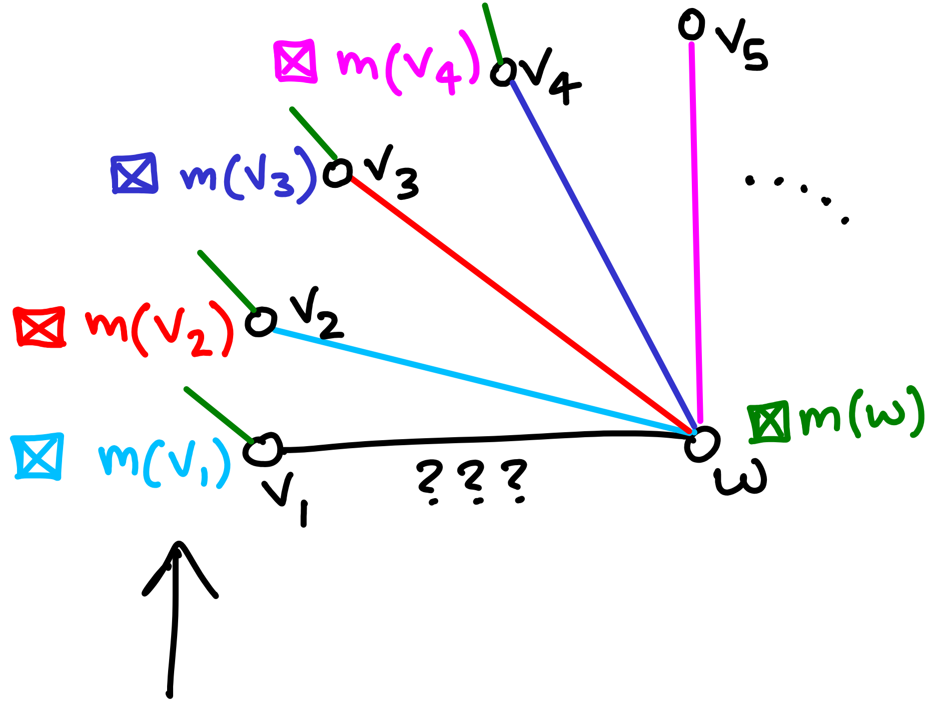
If $\boxtimes m(w)$ DONE

else $\exists\ wv_4$ (thus $\exists\ \boxtimes m(v_3)$)

etc until we reuse a color



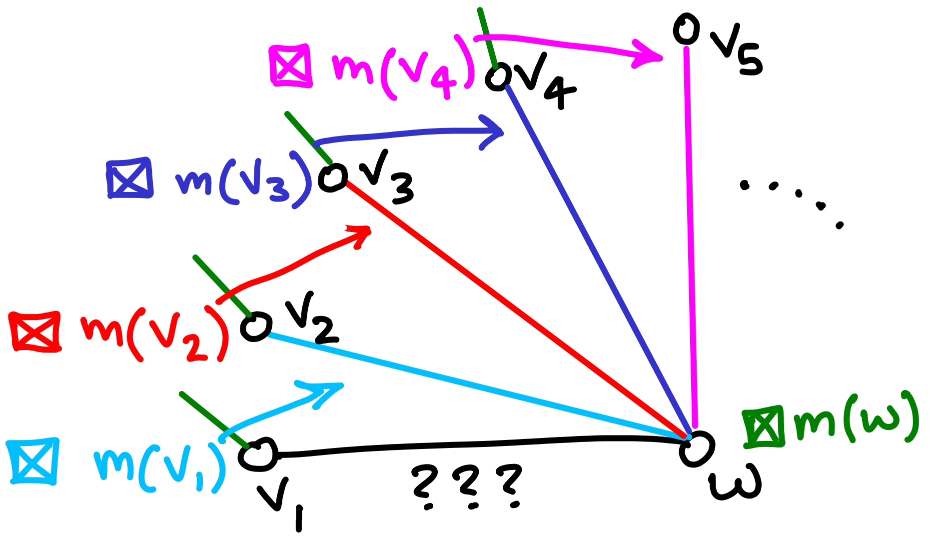
Fan(w) has h neighbors of w: $v_1 \ldots v_h$
& h-1 colors used.



Fan(w) has h neighbors of w: $v_1 \ldots v_h$
& $h-1$ colors used.

for $i=1$ to $h-1$

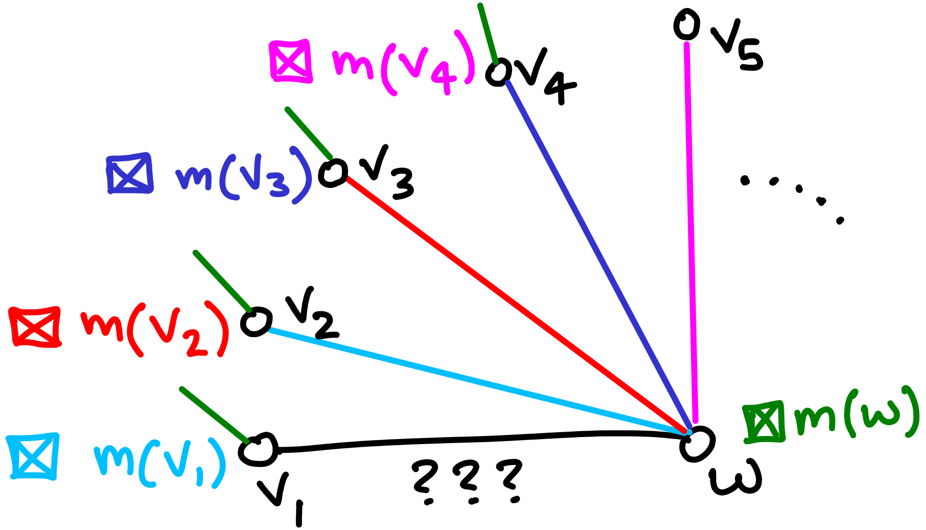
- v_i is missing color $m(v_i) = m_i$



Fan(w) has h neighbors of w: $v_1 \ldots v_h$
& $h-1$ colors used.

for $i=1$ to $h-1$

- v_i is missing color $m(v_i) = m_i$
- $\overline{v_{i+1} w}$ is colored m_i



Fan(w) has h neighbors of w: $v_1 \dots v_h$
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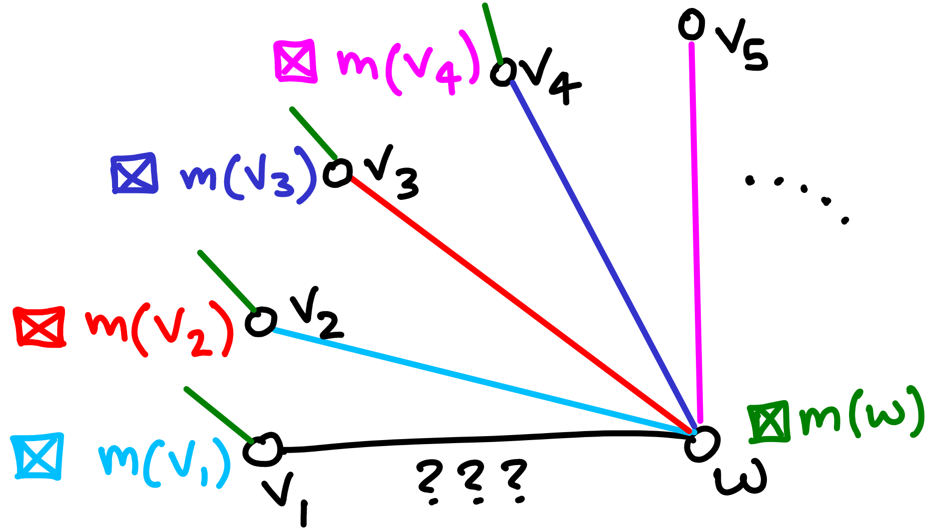
for $i=1$ to $h-1$

- v_i is missing color $m(v_i) = m_i$
- $\overline{v_{i+1}w}$ is colored m_i
- $m_i \neq m(w)$



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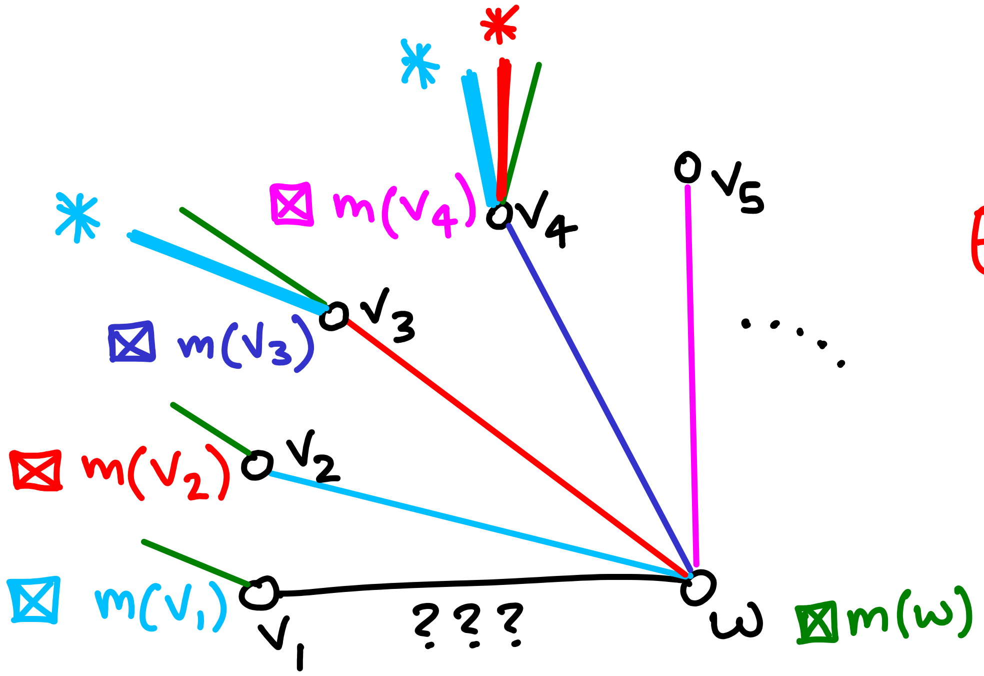
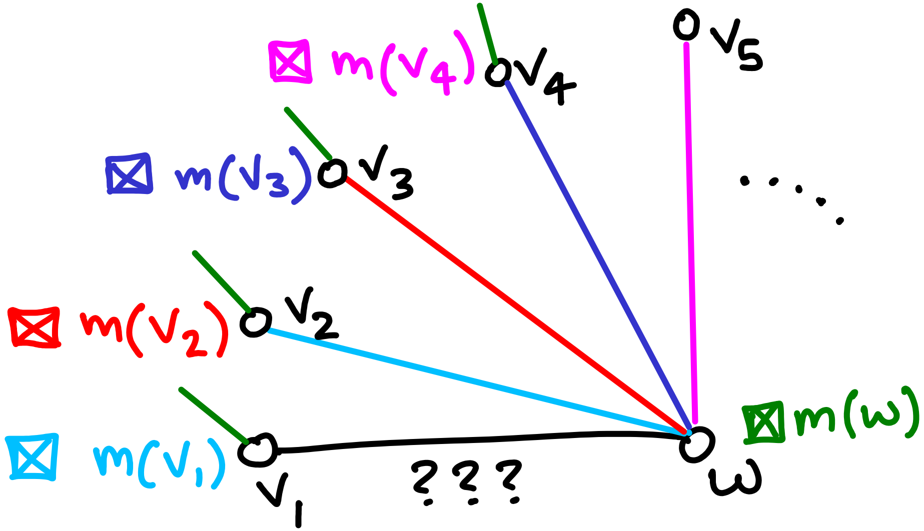


Extra requirement, for $i=2$ to $h-1$

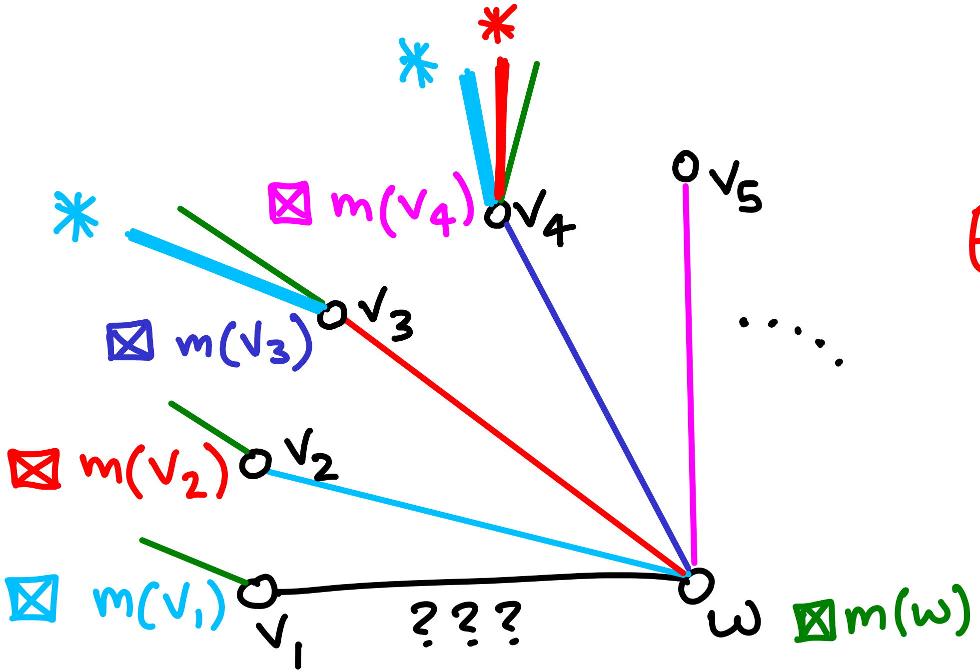
- v_i does not miss any color already missed by v_j $(j<i)$



Fan(w) has h neighbors of w: $v_1 \dots v_h$ & h-1 colors used.

for i=1 to h-1

- v_i is missing color $m(v_i) = m_i$
- $\overline{v_{i+1} w}$ is colored m_i
- $m_i \neq m(w)$

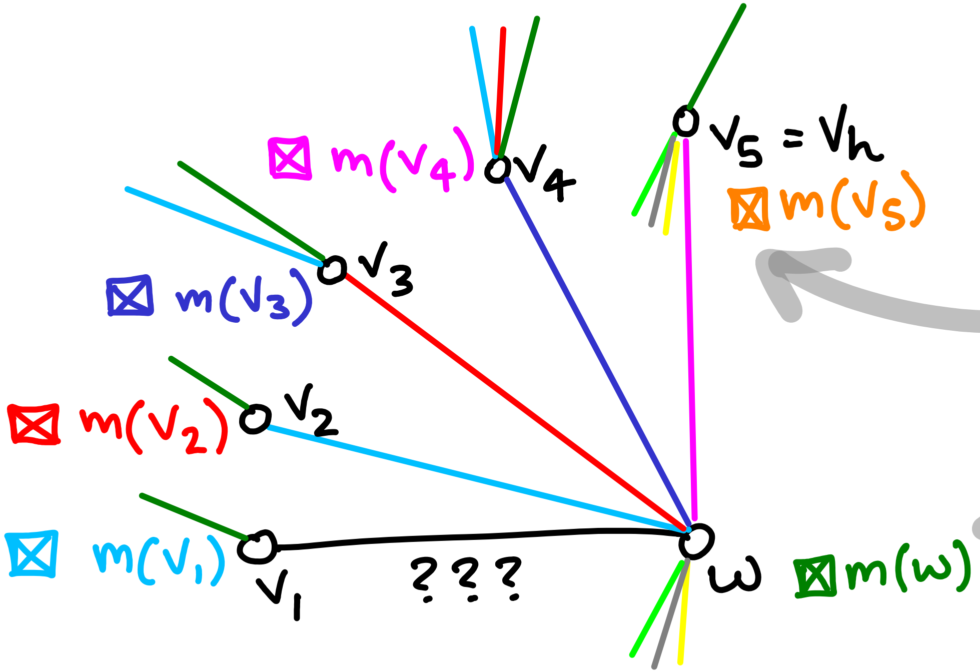


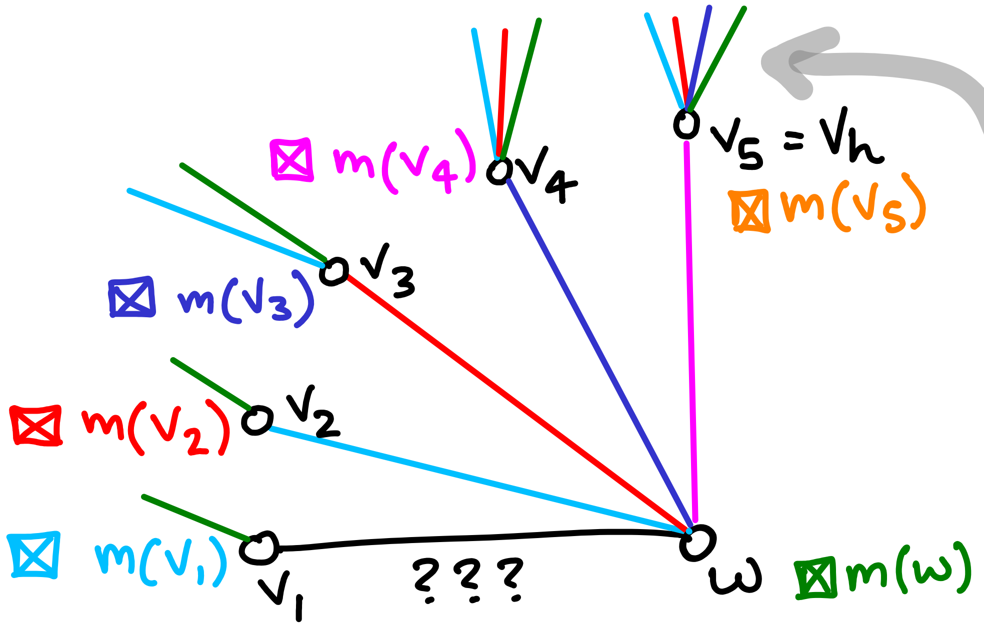
Extra requirement, for i=2 to h-1

- v_i does not miss any color already missed by v_j $(j<i)$

Our algo starts w/ a fan w/ $h \geqslant 3$

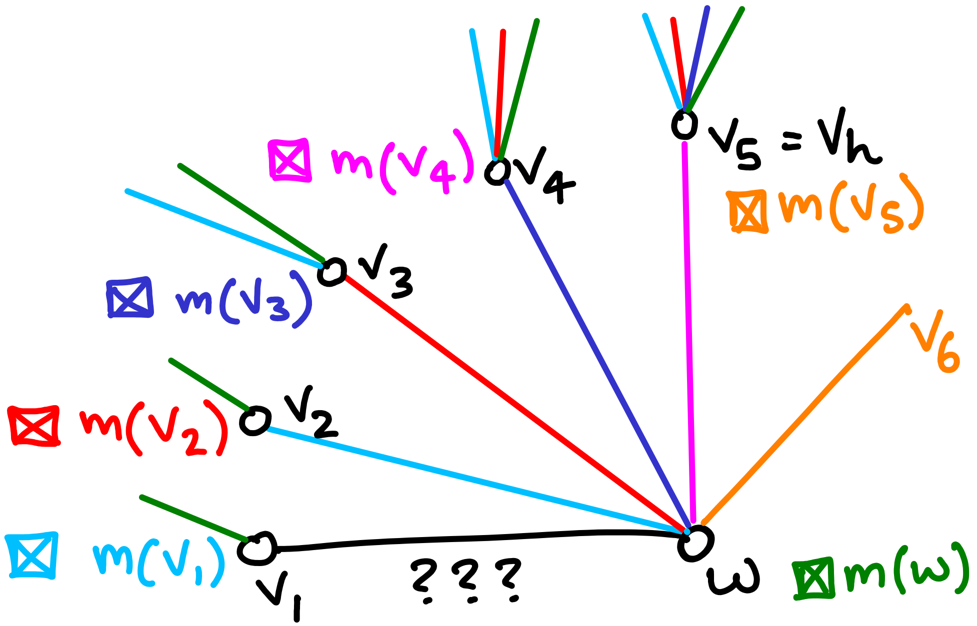
If v_h has no $m_h = m_w$



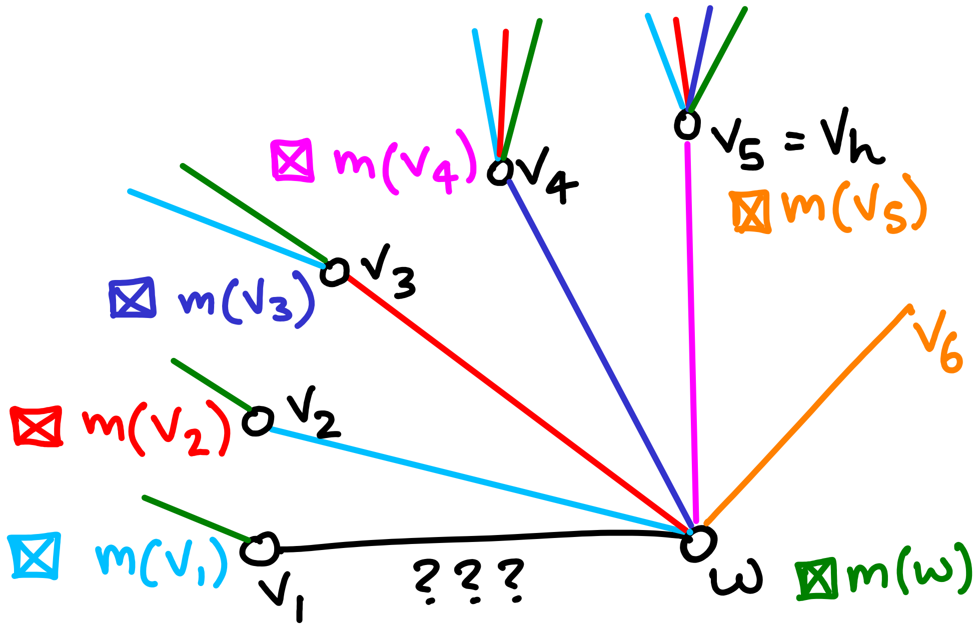


If v_h has no $m_h = m_w$ & all previous colors on fan are incident,

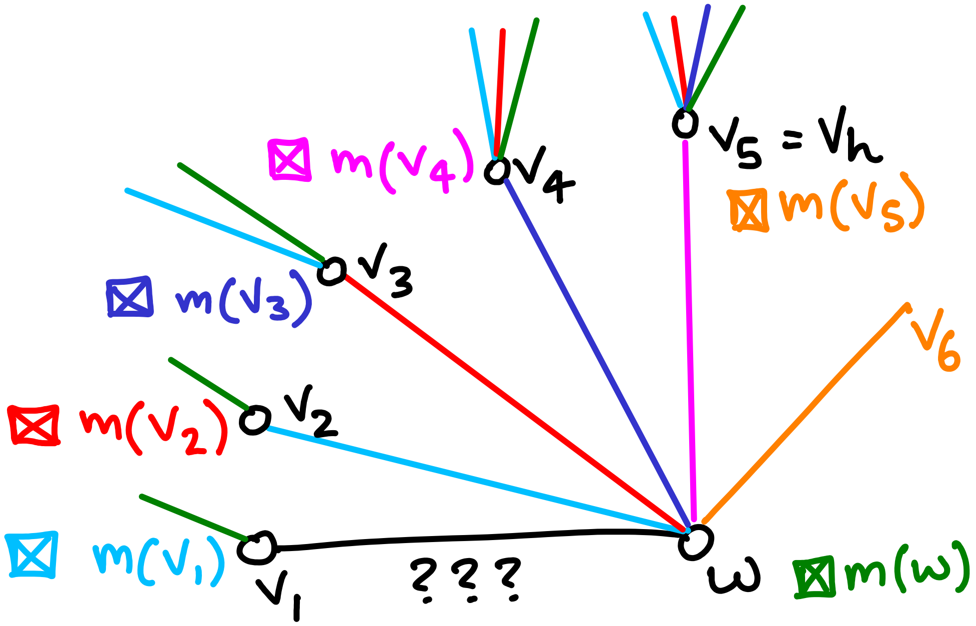
(i.e. no common missing colors)



If v_h has no $m_h = m_w$ & all previous colors on fan are incident, then there must be a v_6



If v_h has no $m_h = m_w$ &
all previous colors on fan are incident,
then there must be a v_6
but this can't continue forever.
[so one of two things happens]
(2 conditions set above)



If v_h has no $m_h = m_w$ &
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[so one of two things happens]

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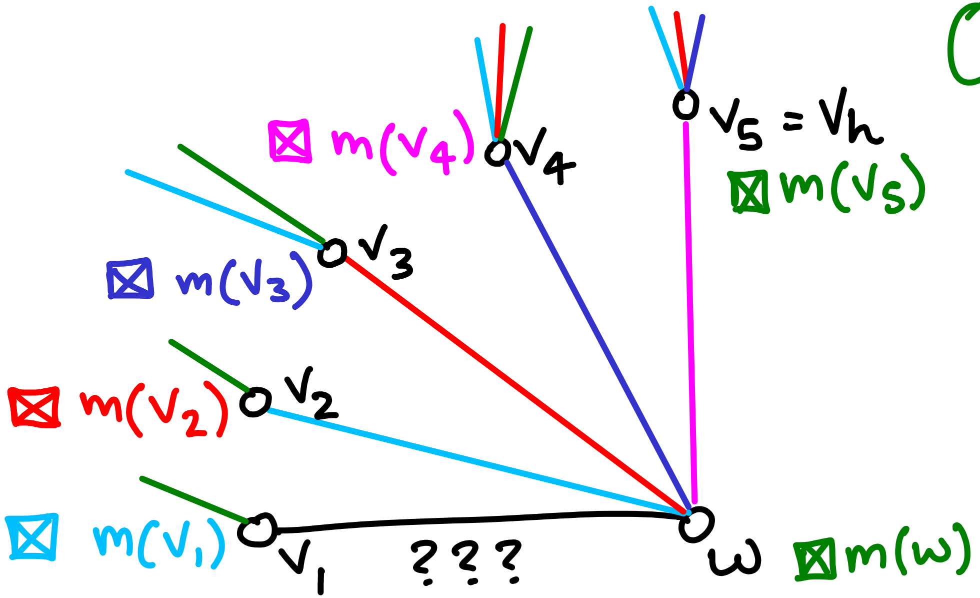

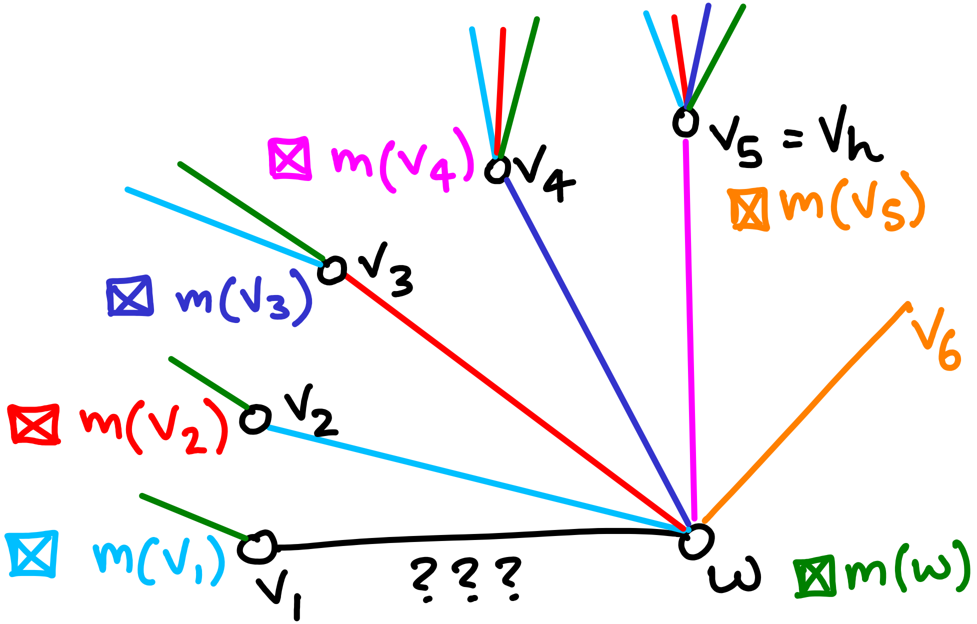

If v_h has no $m_h = m_w$ &
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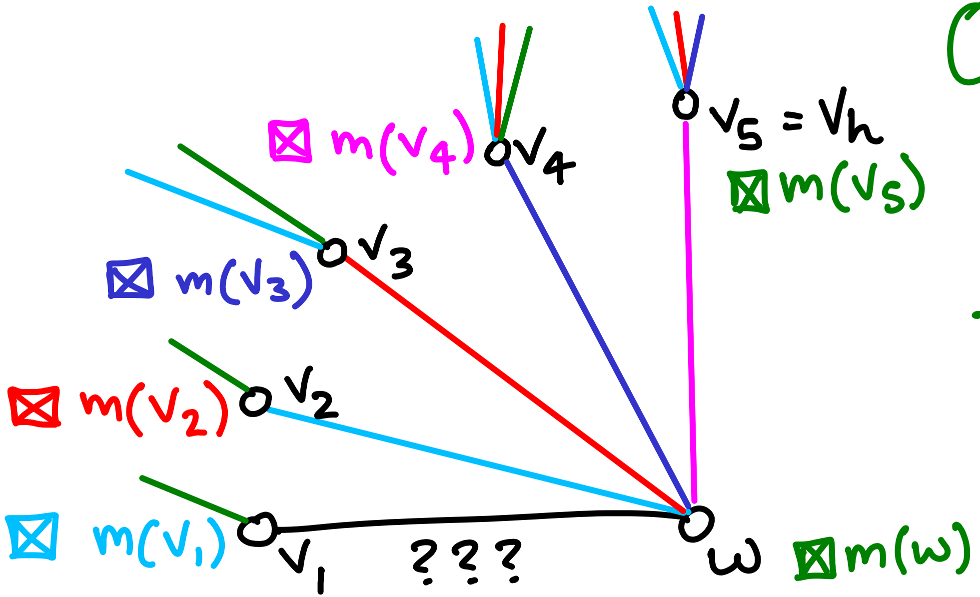

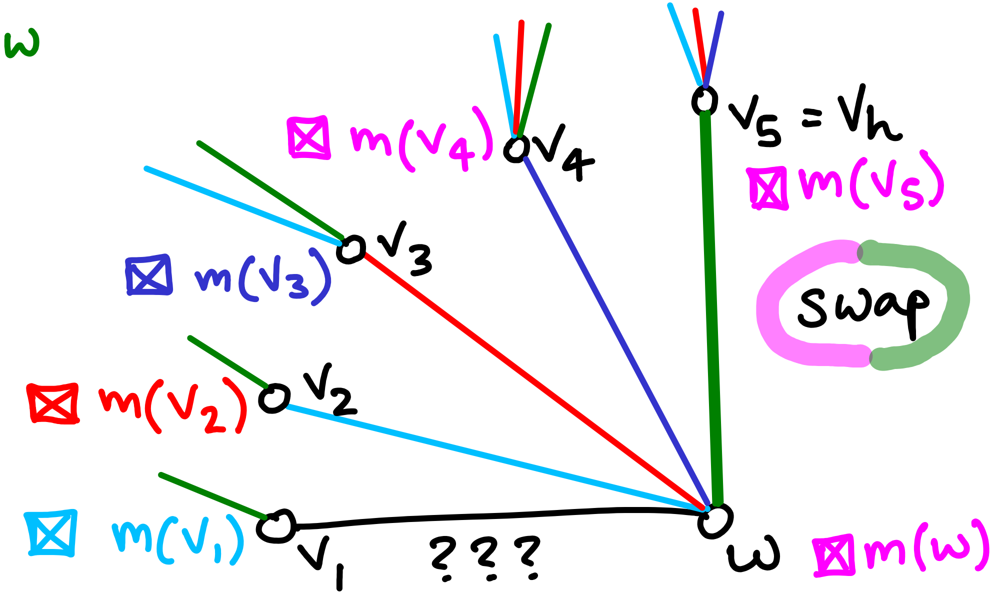

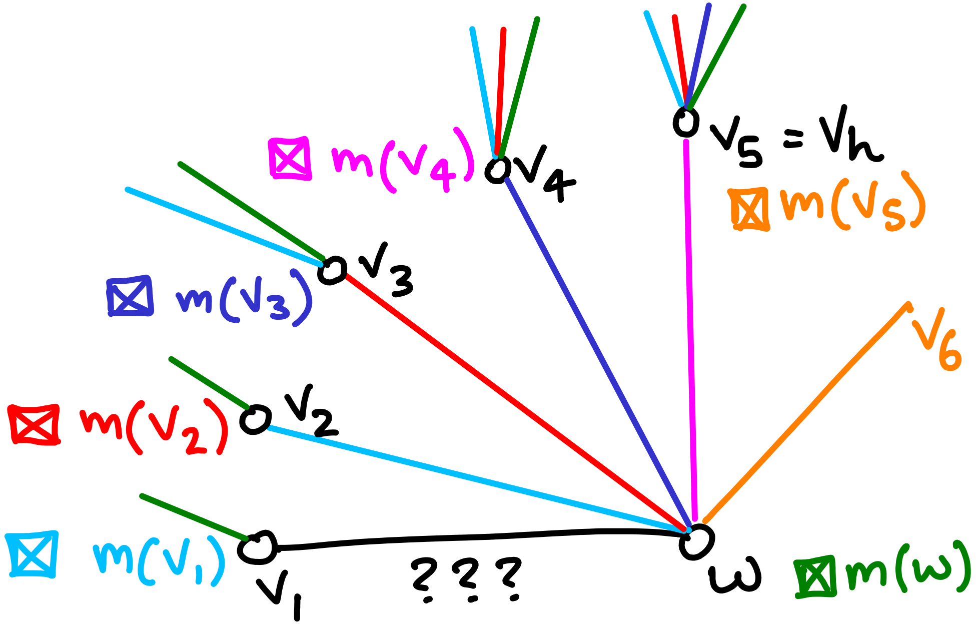

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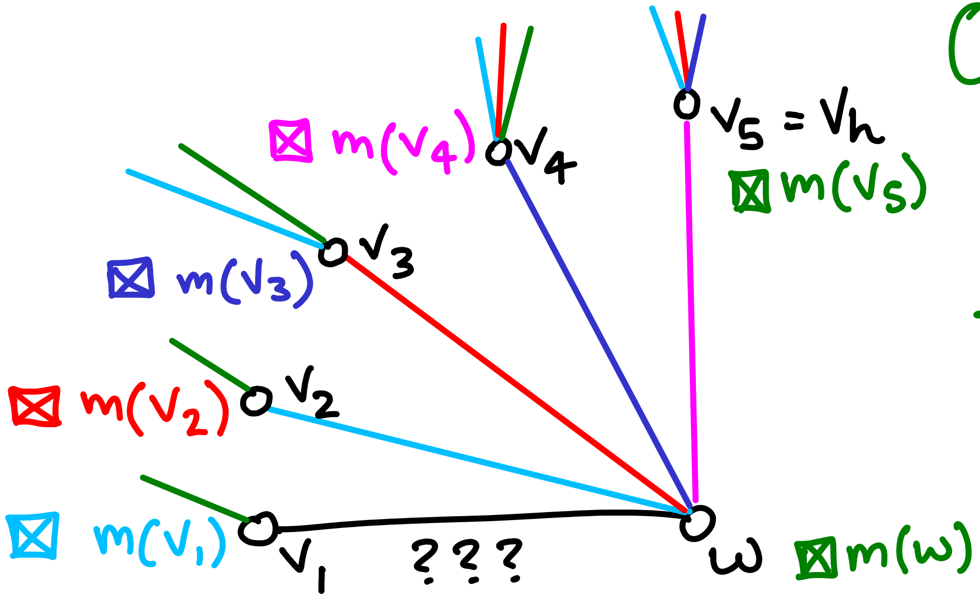

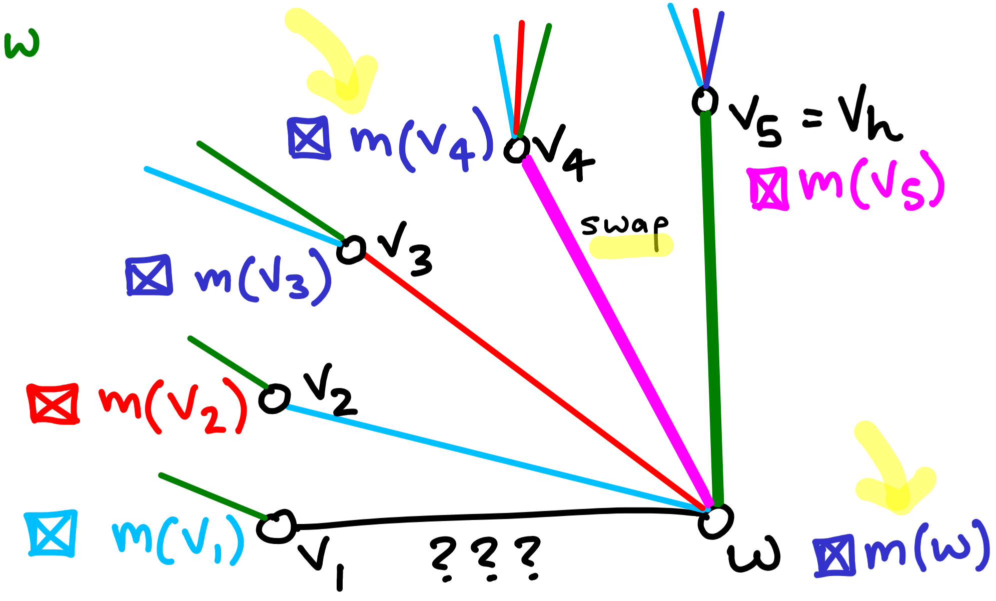

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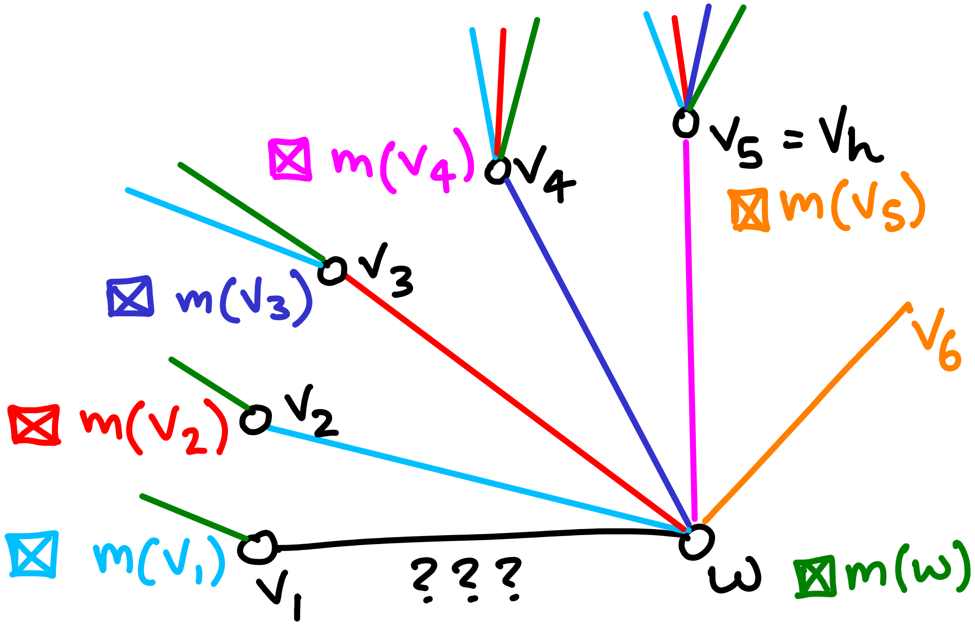

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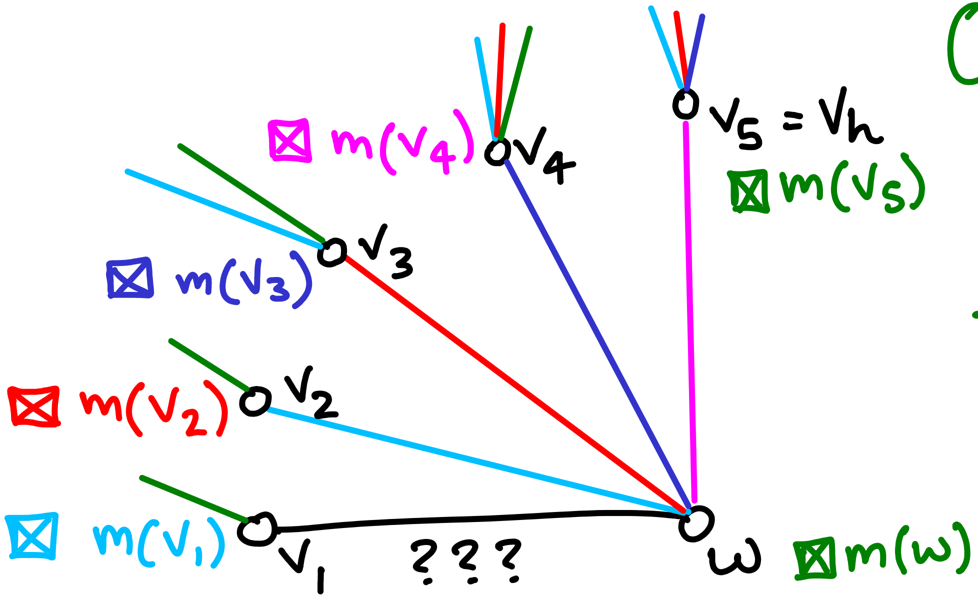

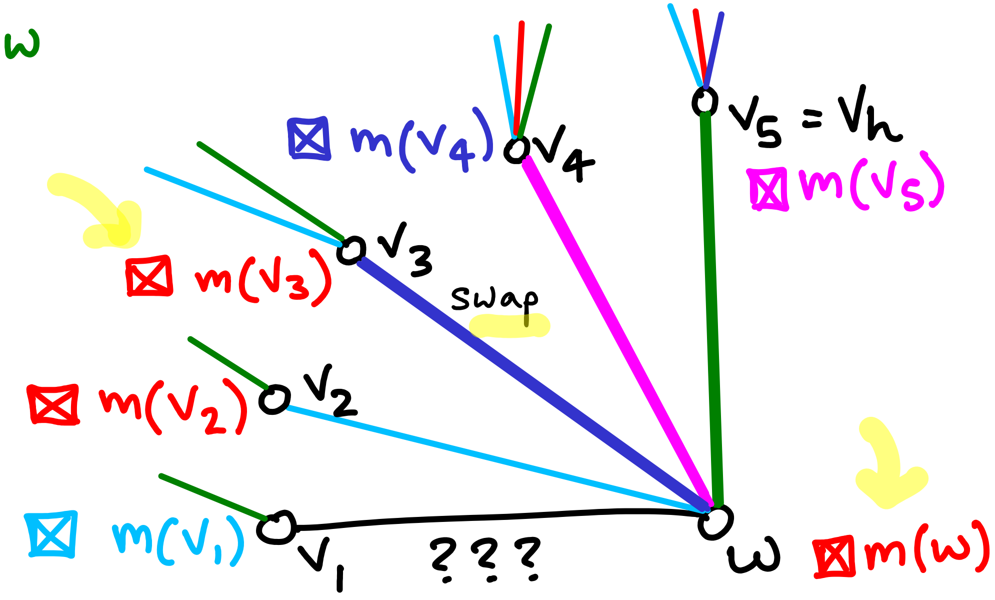

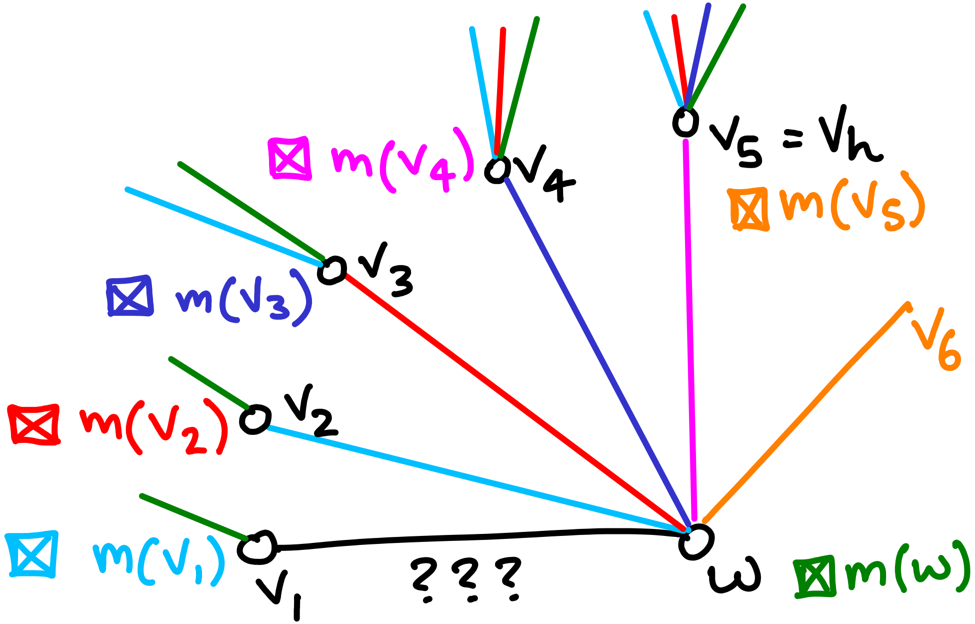

If v_h has no $m_h = m_w$ & all previous colors on fan are incident, then there must be a v_6 but this can't continue forever.

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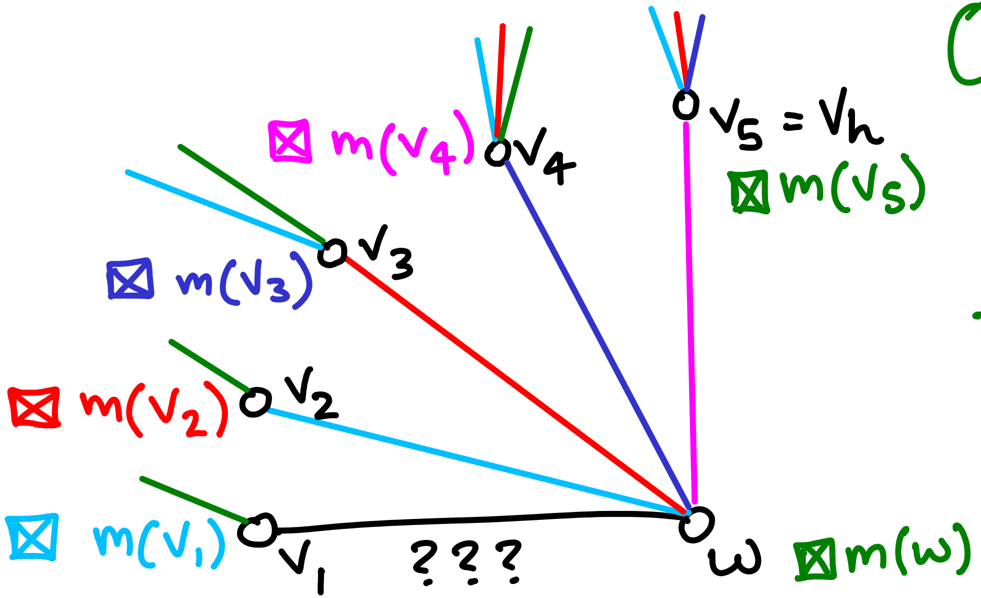

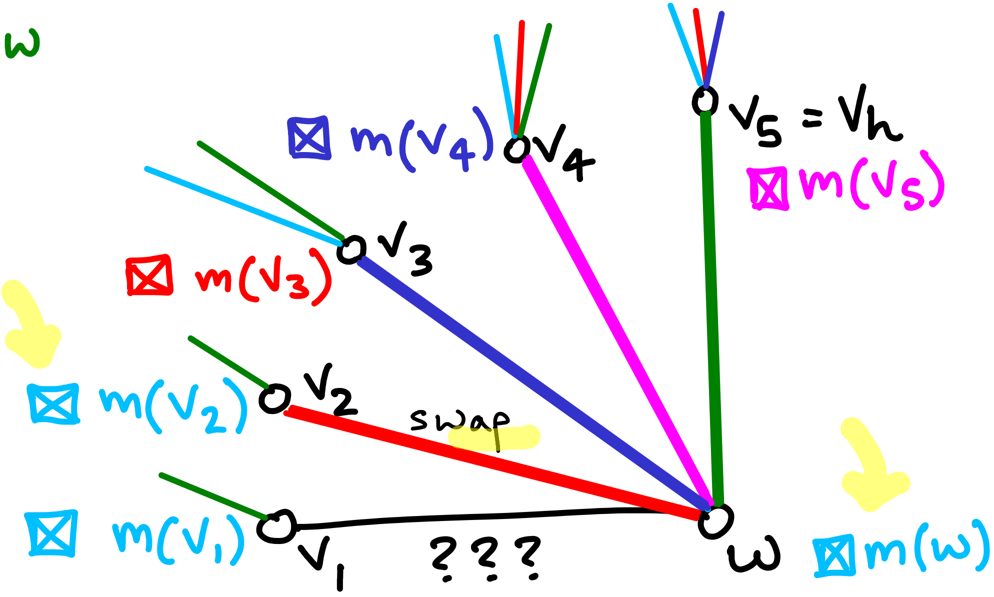

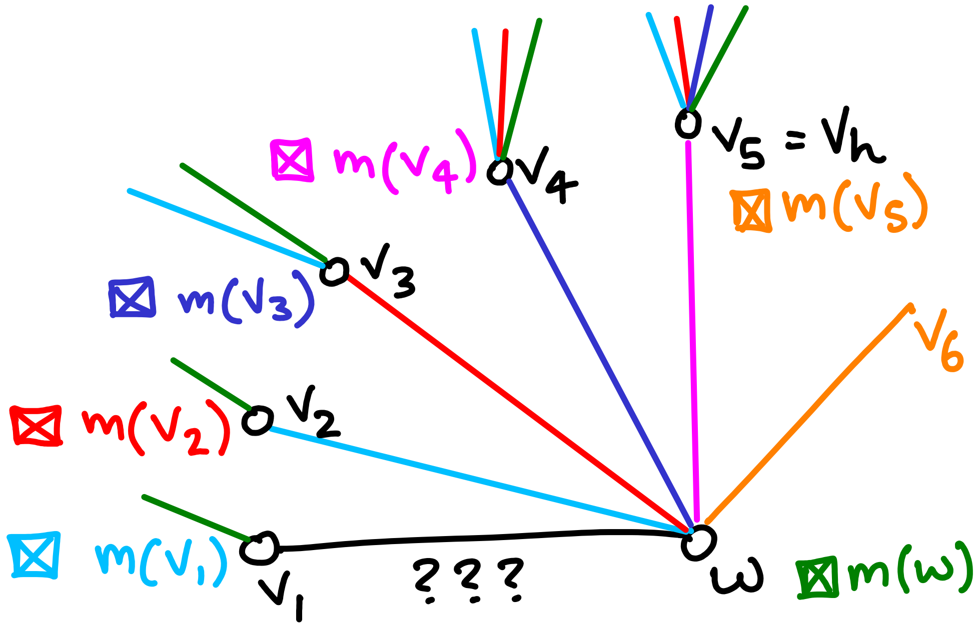

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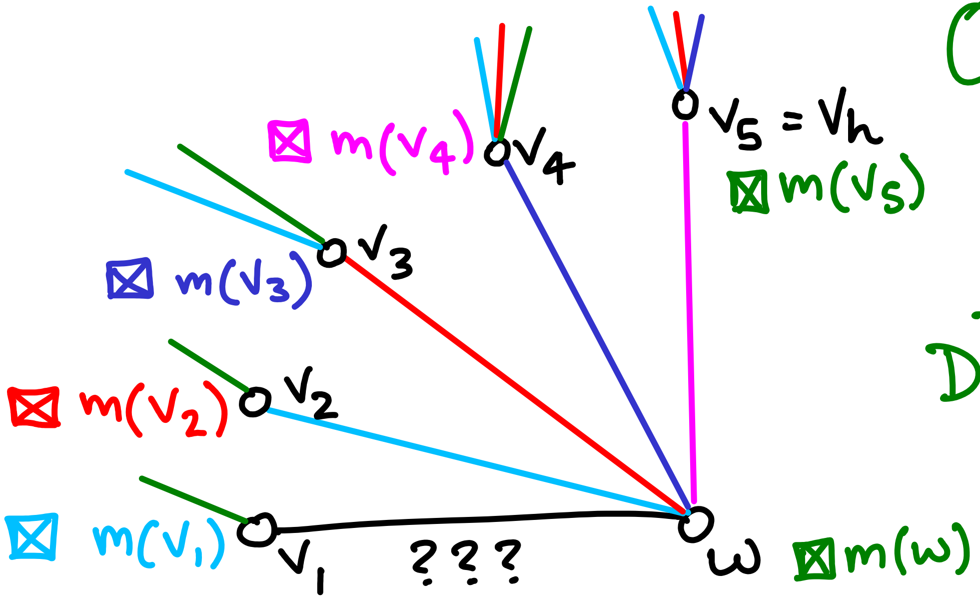
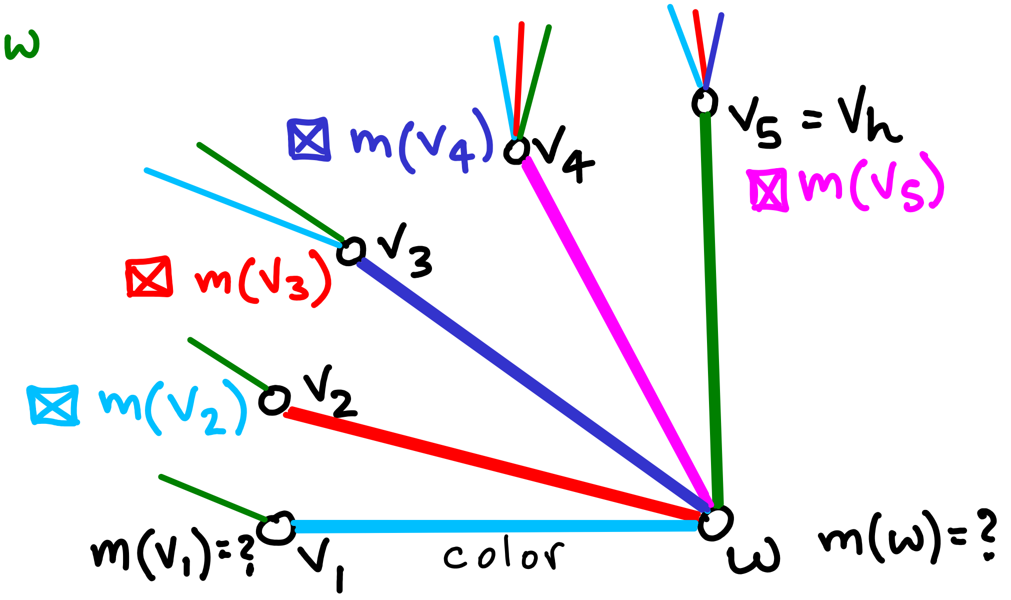


DONE



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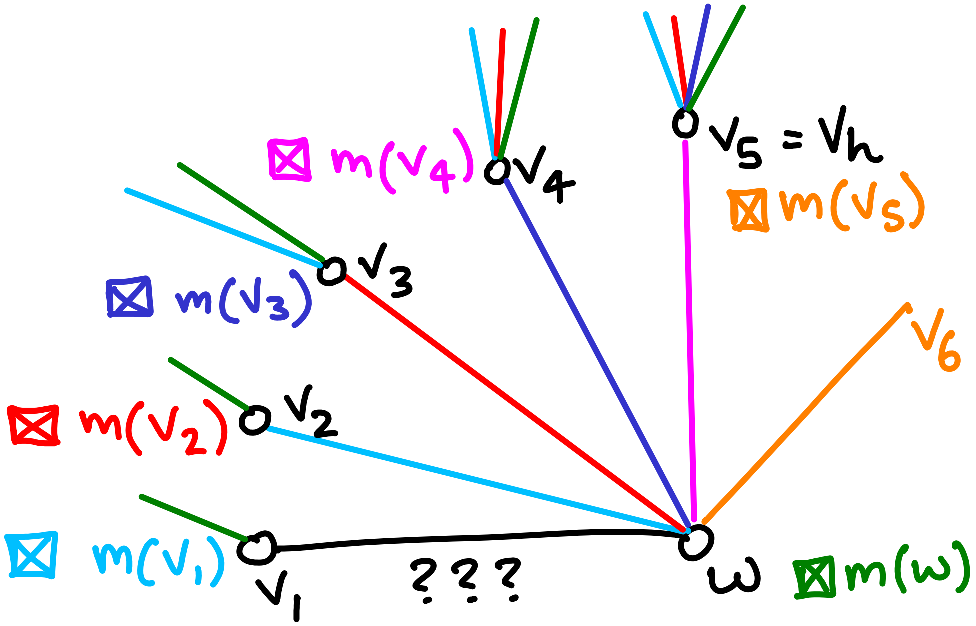
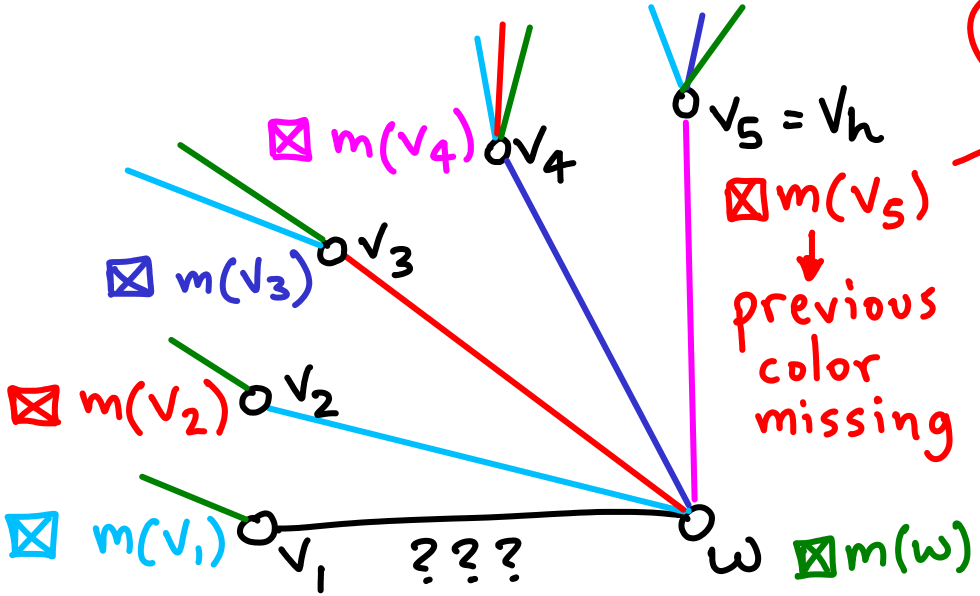
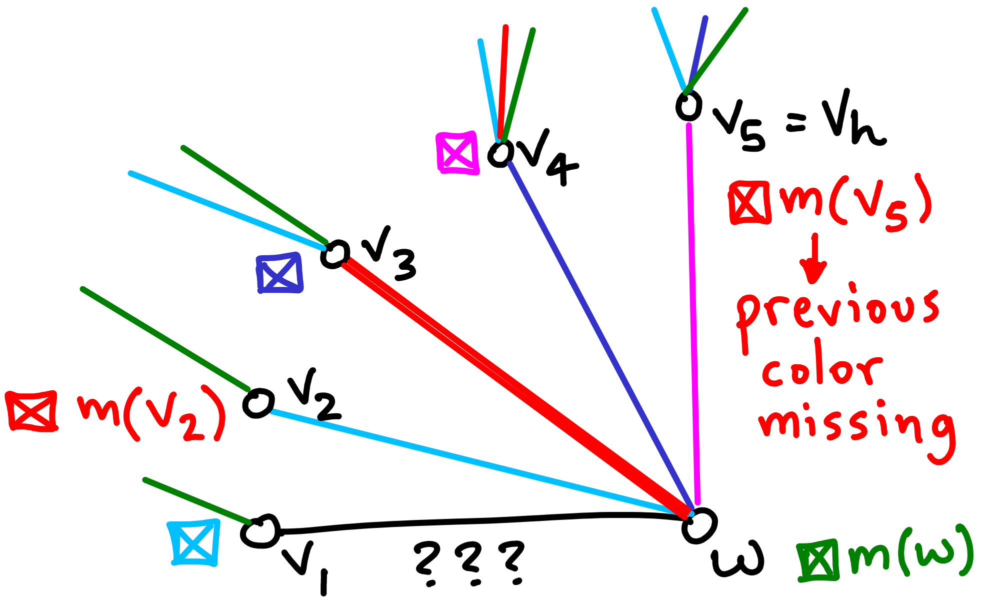


2

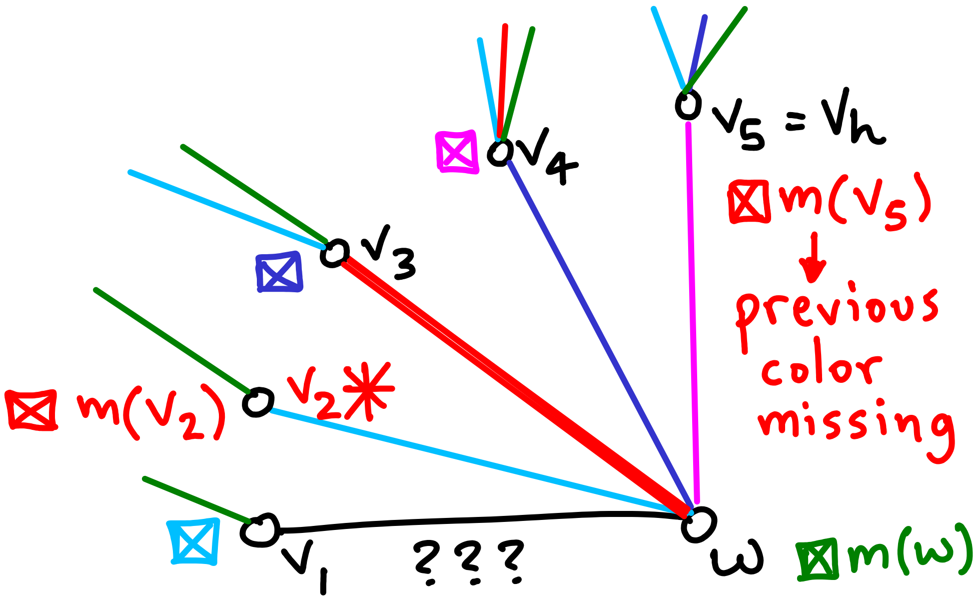


$\Rightarrow m_h = m_{(v_k)}$ for some $1 \leq k < h$

v_h (= v_5) is missing a previously used color



If v_h ($=v_5$) is missing a previously used color, on $v_i w$ ($= v_3 w$)
then $\exists$ v_{i-1} (v_2) also missing that color, i.e. $m_h = m_{i-1}$



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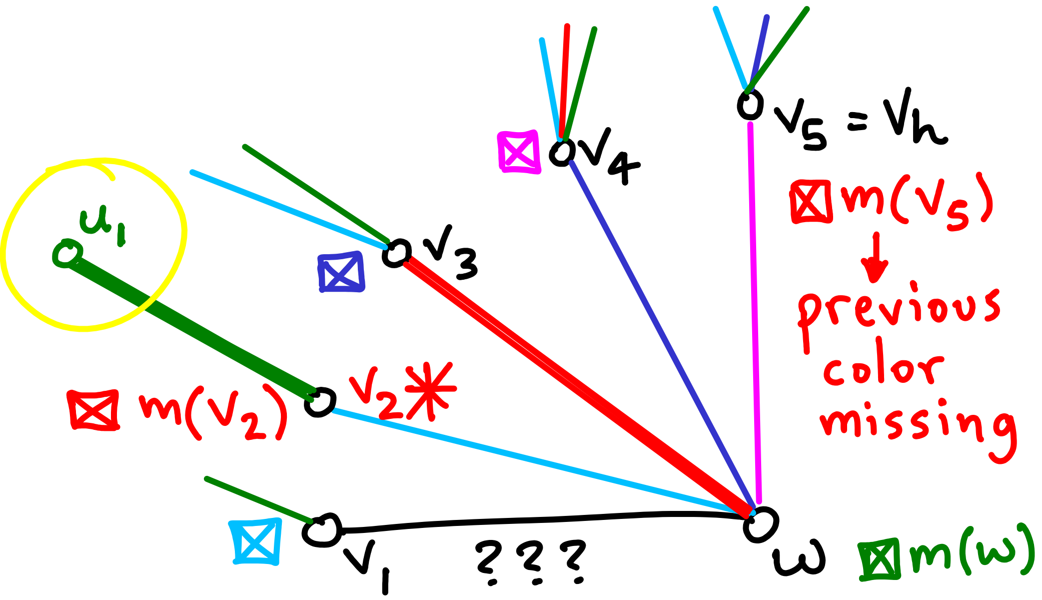
We also know that v_{i-1} has an edge of color $m(w)$ [as do all v_j]
Let this edge lead to u_1.



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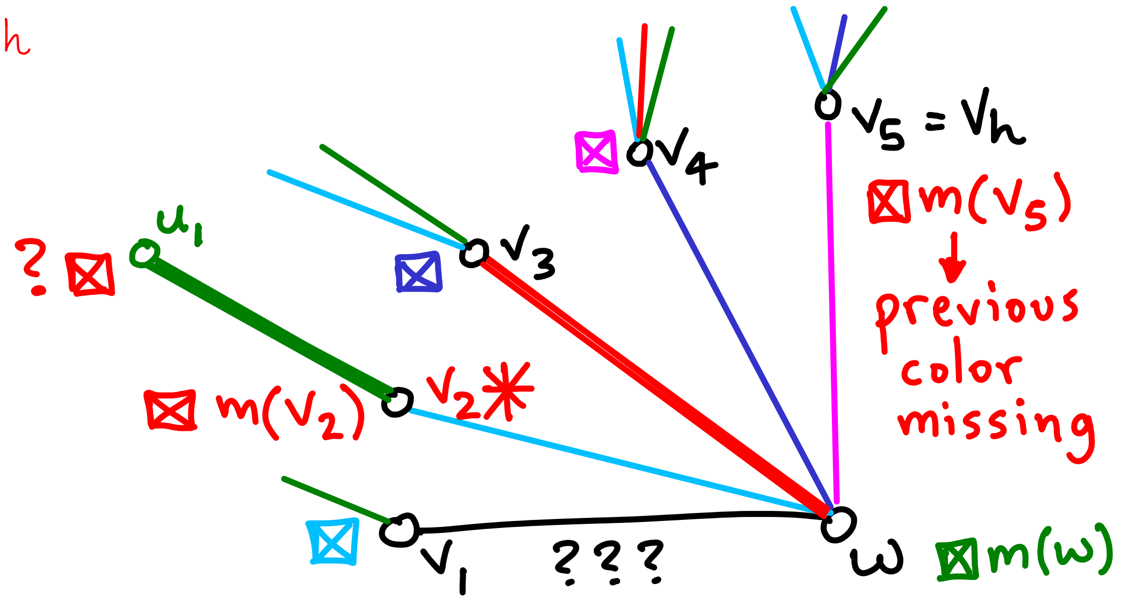
What happens if u_1 misses m_{i-1} ?

i.e m_h



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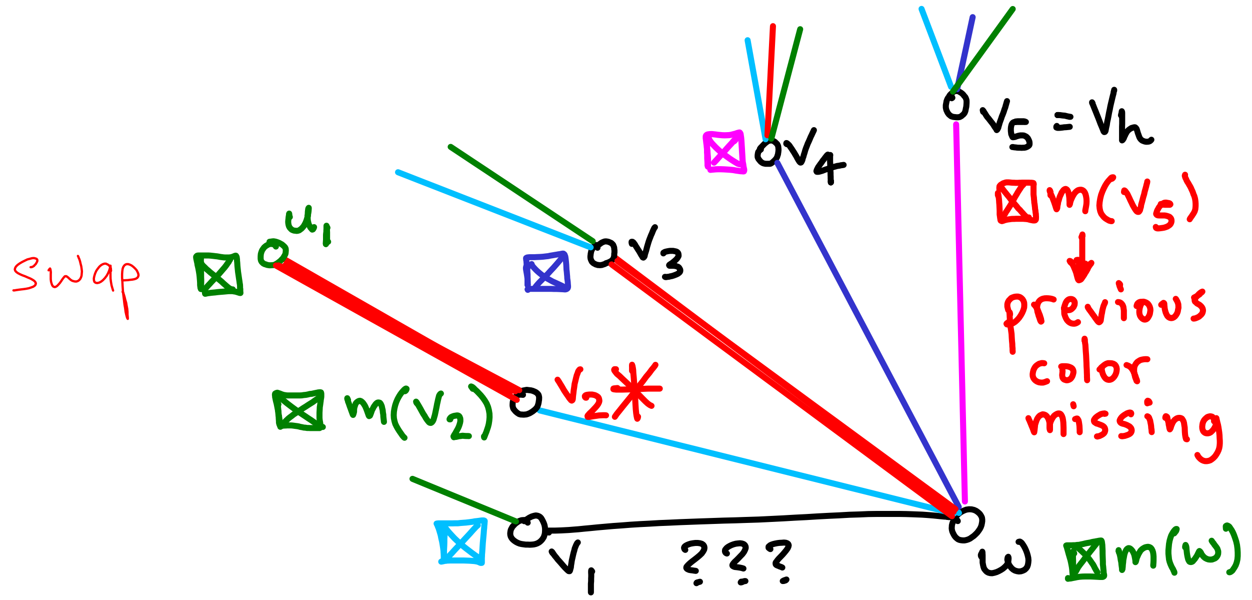

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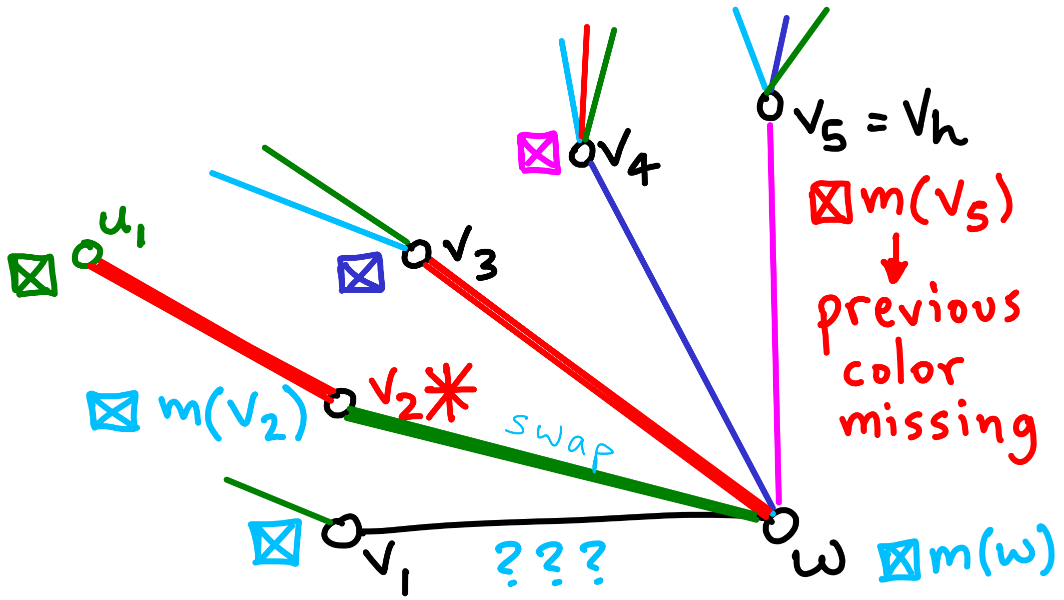
Let this edge lead to u_1.

What happens if u_1 misses m_{i-1} ?

↓

We get a chain reaction &

color $v_1 w = m(v_1)$



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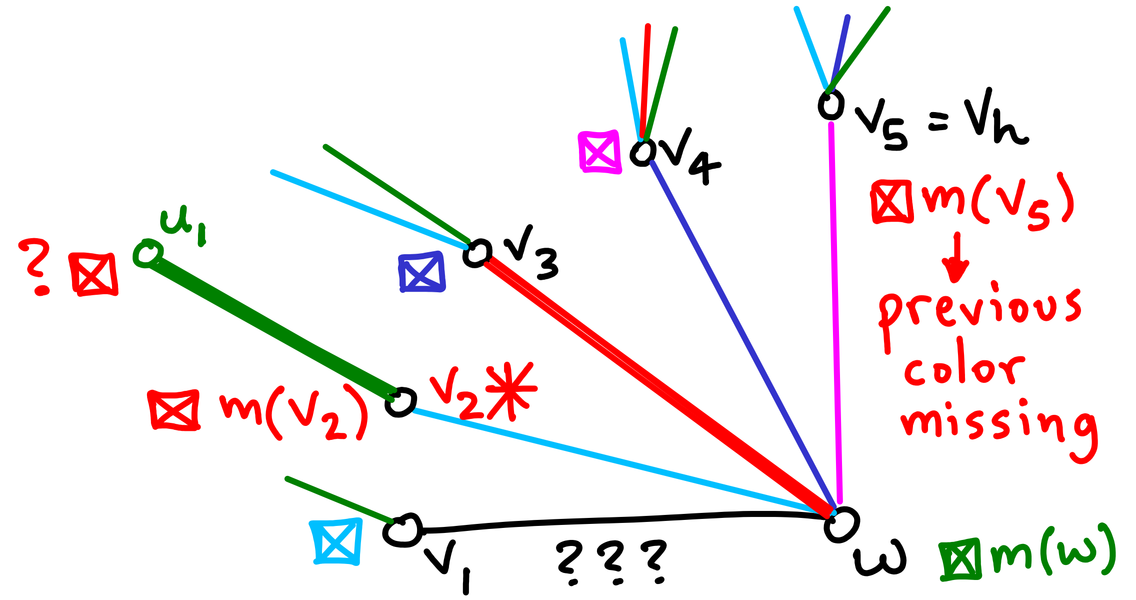
Let this edge lead to u_1.

What happens if u_1 misses m_{i-1} ?
↳ We would be DONE



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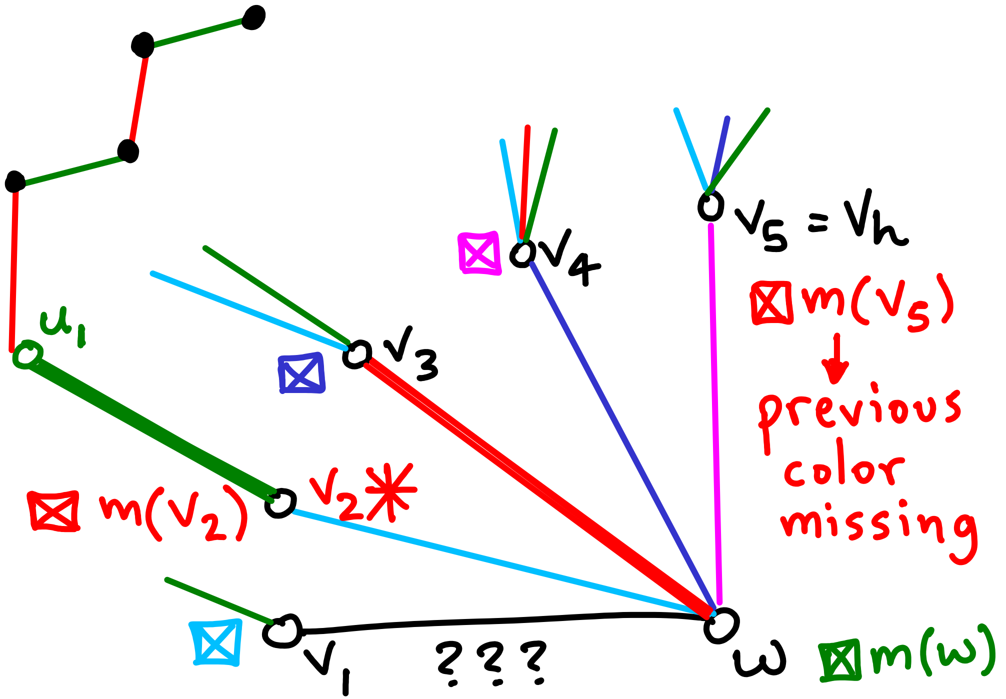
So we get a path v_2 ... }

↳ not a cycle: can't return to v_2 or anywhere on path.



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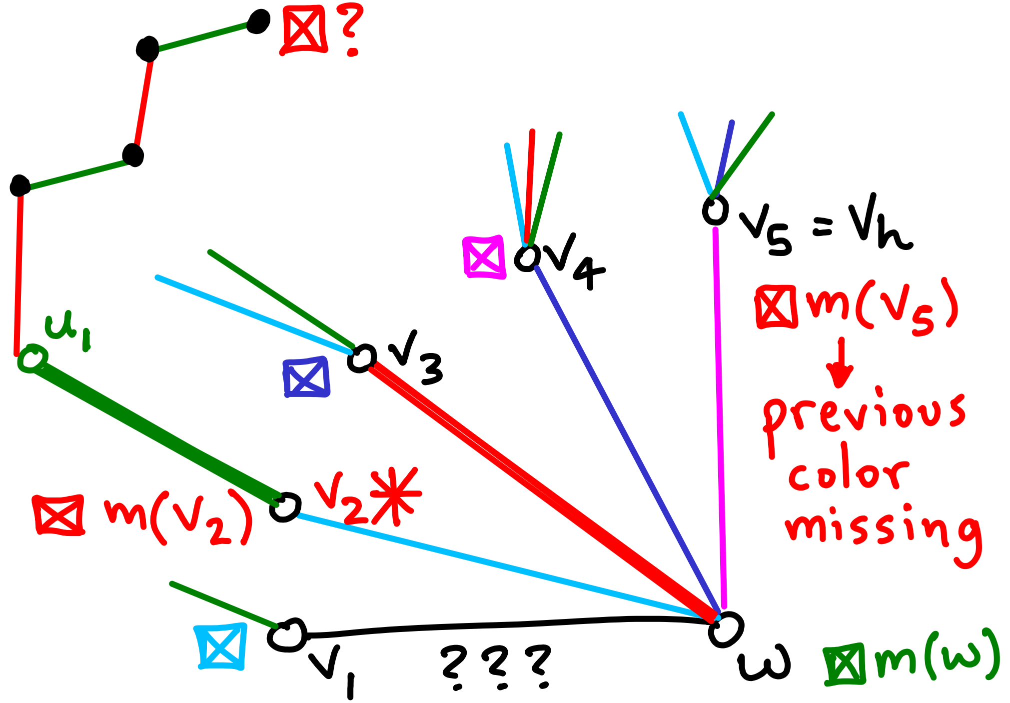
So we get a path v_2 ... }
↳ not a cycle: can't return to v_2 or anywhere on path.

If the path ends w/o touching w...



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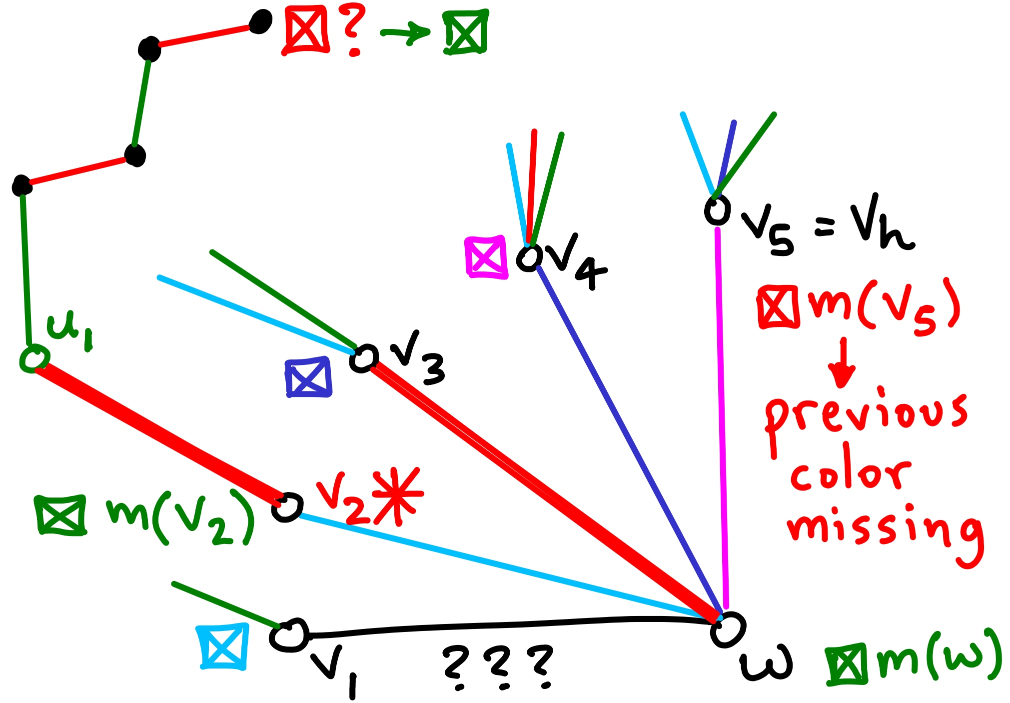
So we get a path v_2 •/\/\/\ ... }
↳ not a cycle: can't return to v_2 or anywhere on path.

If the path ends w/o touching w
we will be DONE as before



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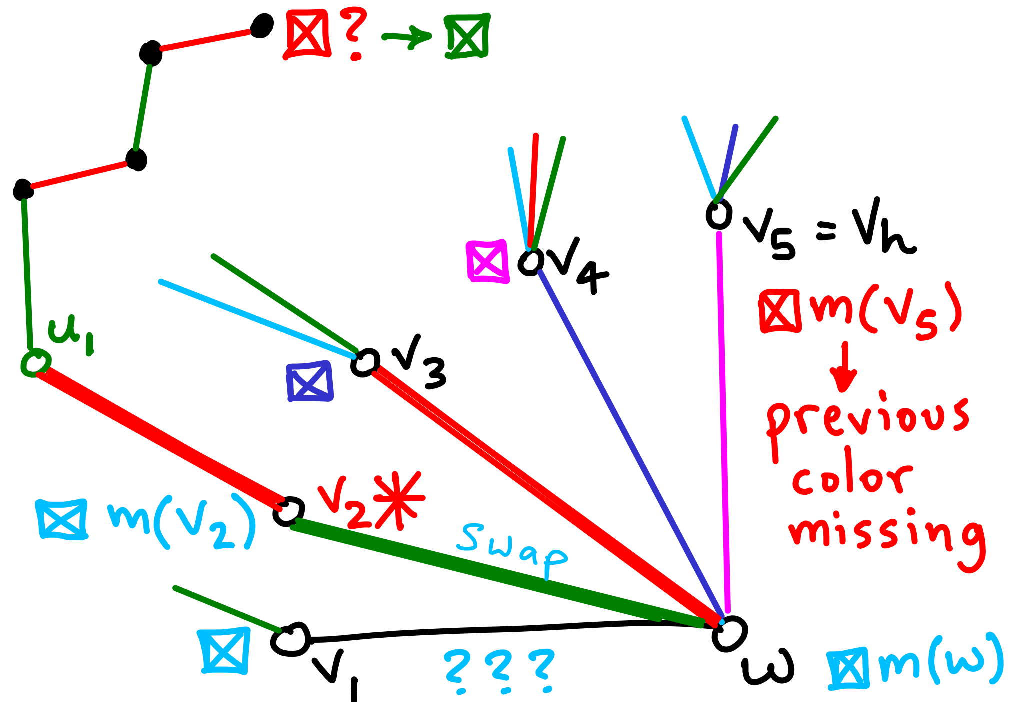
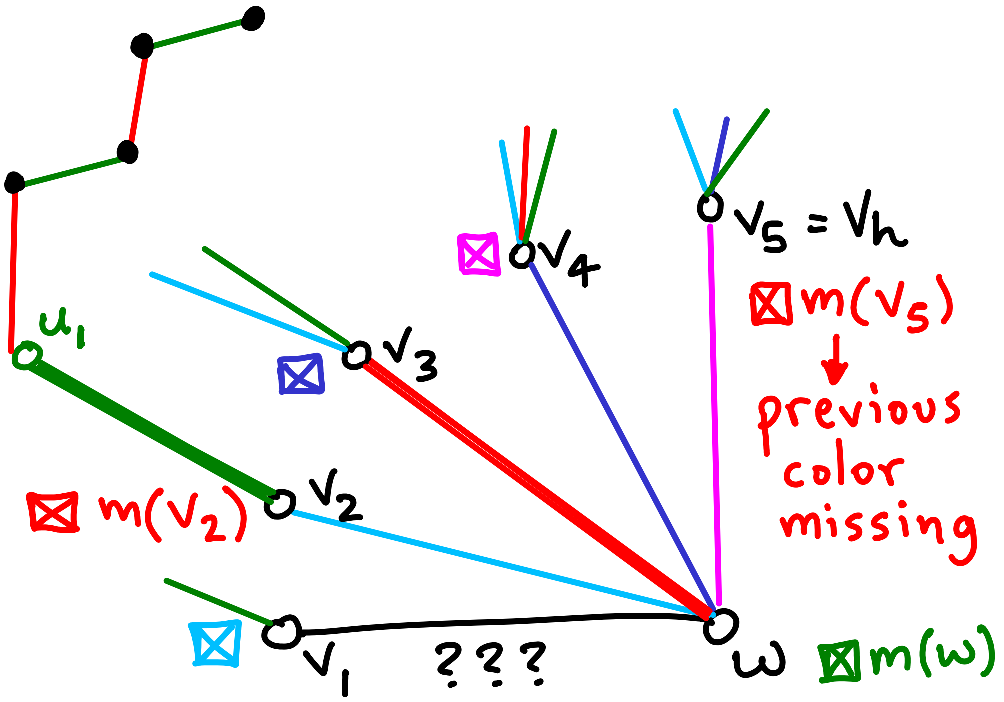


Final case: path ends at w. (it can't go through w because $\boxtimes m(w)$)



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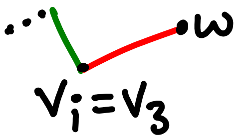
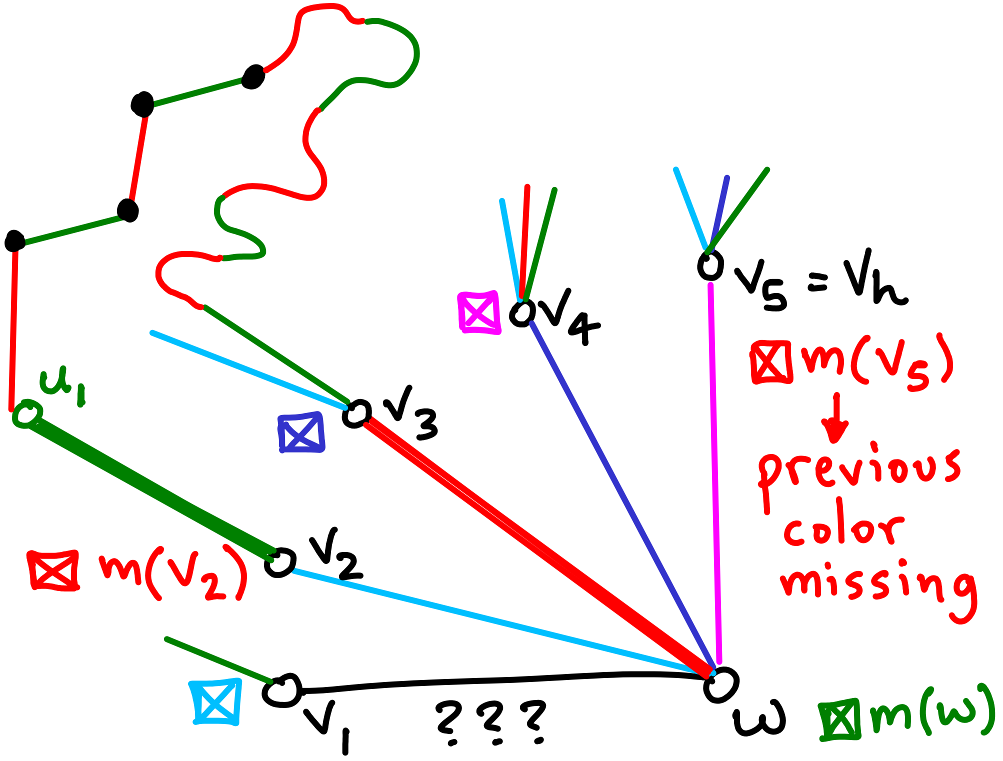
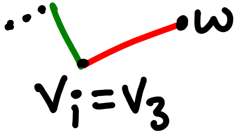
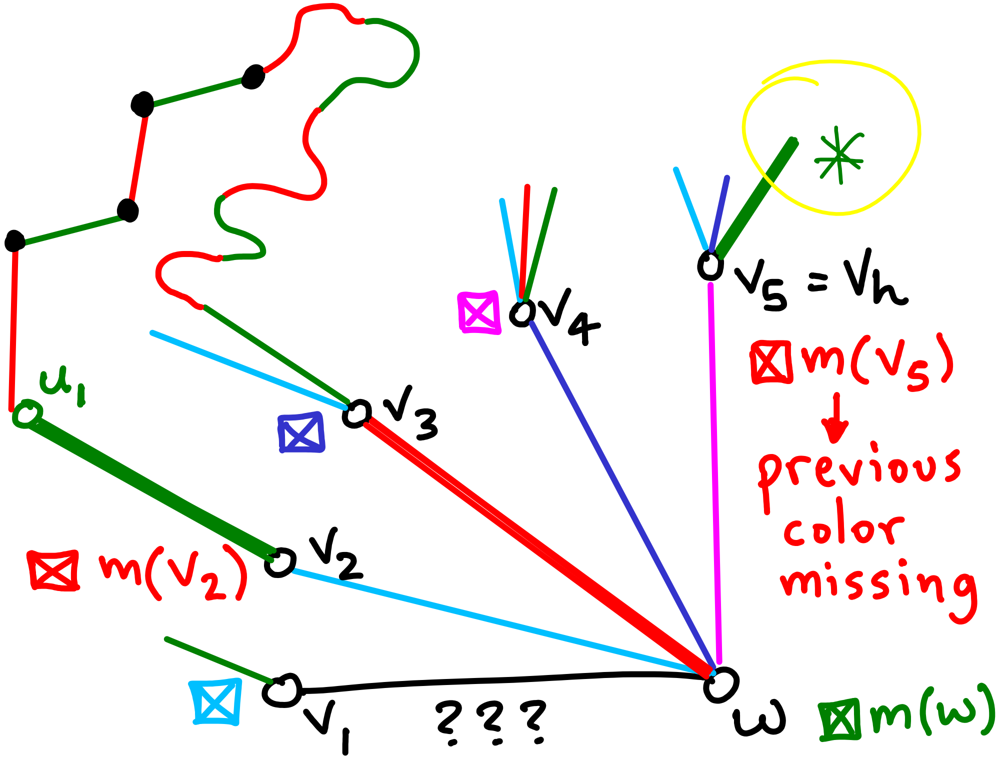
So path ends w/ 



Final case: path ends at w. (it can't go through w because ⊠ $m(w)$)

So path ends w/ 

Now look at edge from v_h with color $m(w)$



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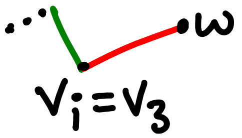
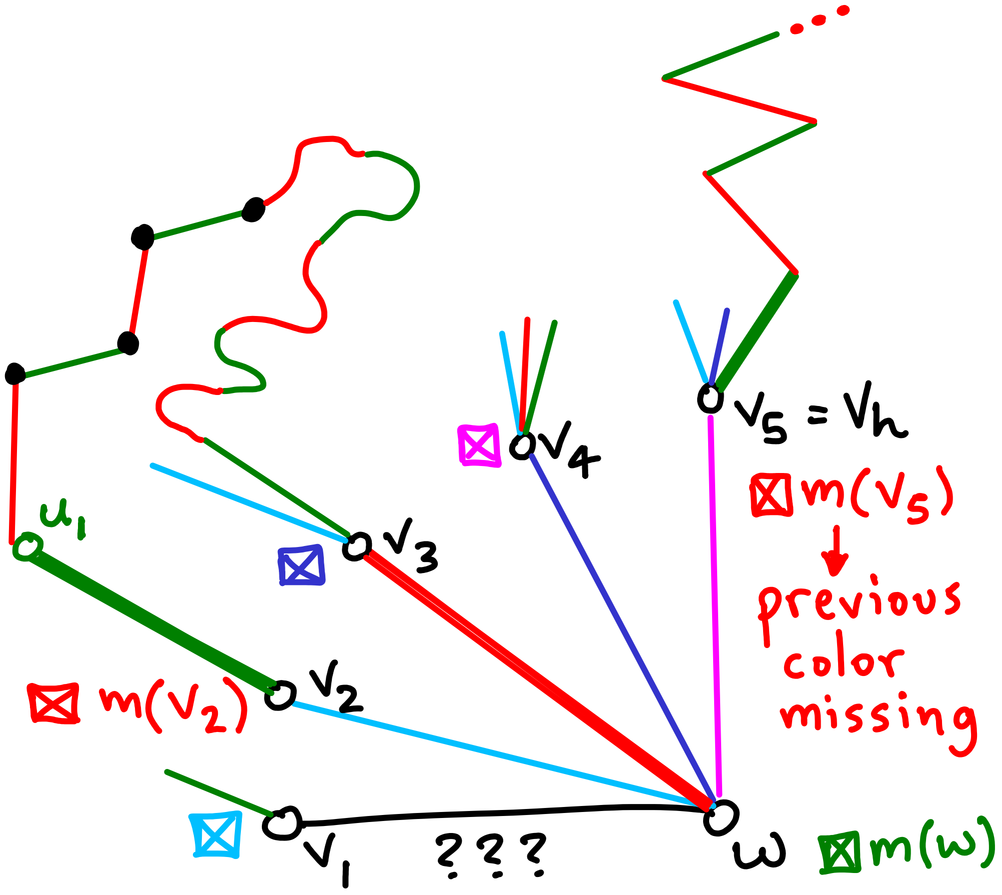
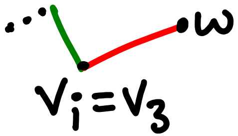
So path ends w/ 

Now look at edge from v_h with color $m(w)$

Extend to a path with same colors as previous path.



Final case: path ends at w. (it can't go through w because ⊠ $m(w)$)

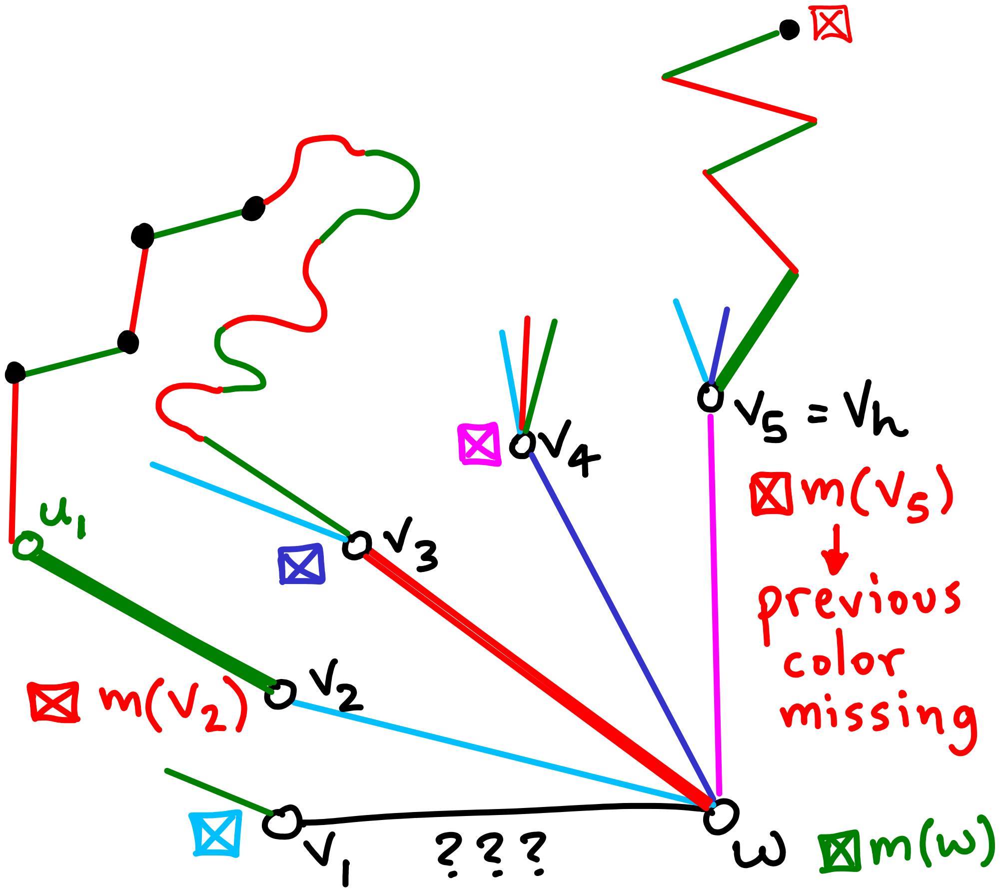
So path ends w/ 

Now look at edge from v_h with color $m(w)$

Extend to a path with same colors as previous path.

↳ must be simple (no cycles)
& <u>can't use w at all</u>

(actually it can't touch previous path)



Final case: path ends at w. (it can't go through w because $\boxtimes m(w)$)

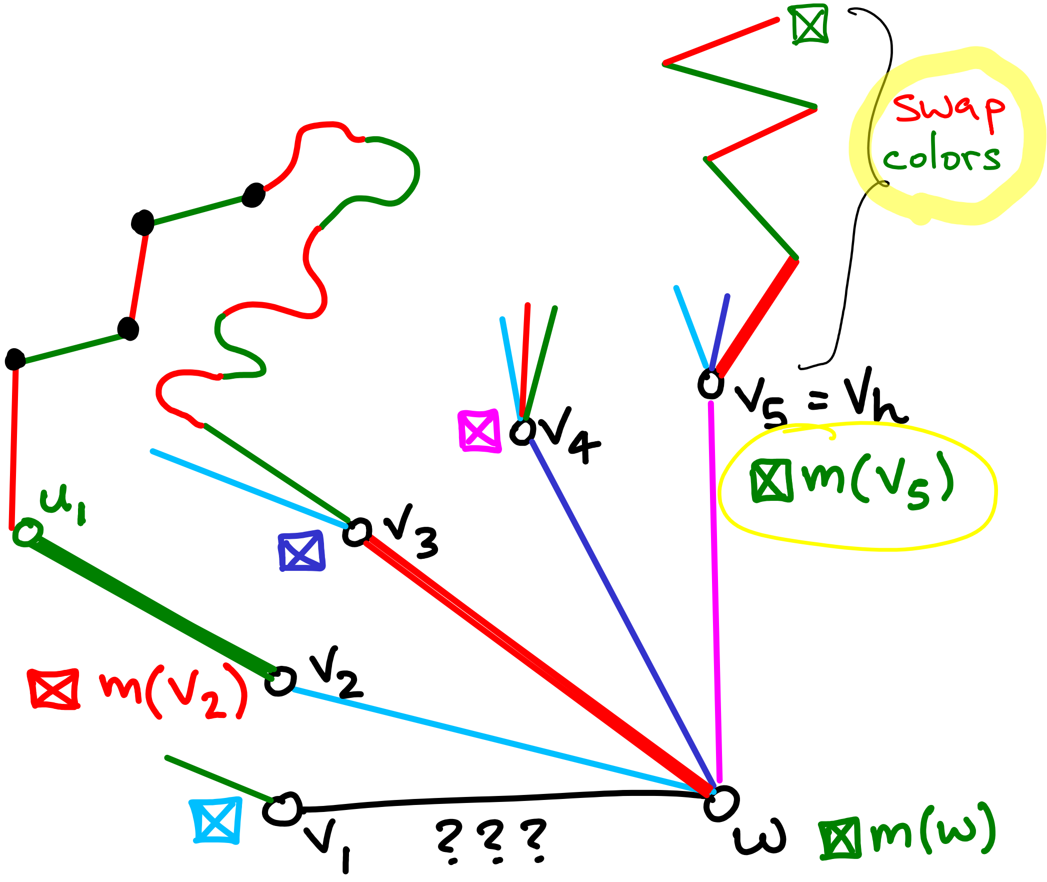
So path ends w/ $\cdots$ $v_i = v_3$ $\bullet w$

Now look at edge from v_h with color $m(w)$

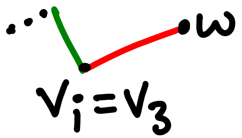
Extend to a path with same colors as previous path.

↳ must be simple (no cycles) & <u>can't use w at all</u>

Then swap colors on the v_h path



Final case: path ends at w. (it can't go through w because $\boxtimes m(w)$)

So path ends w/ 

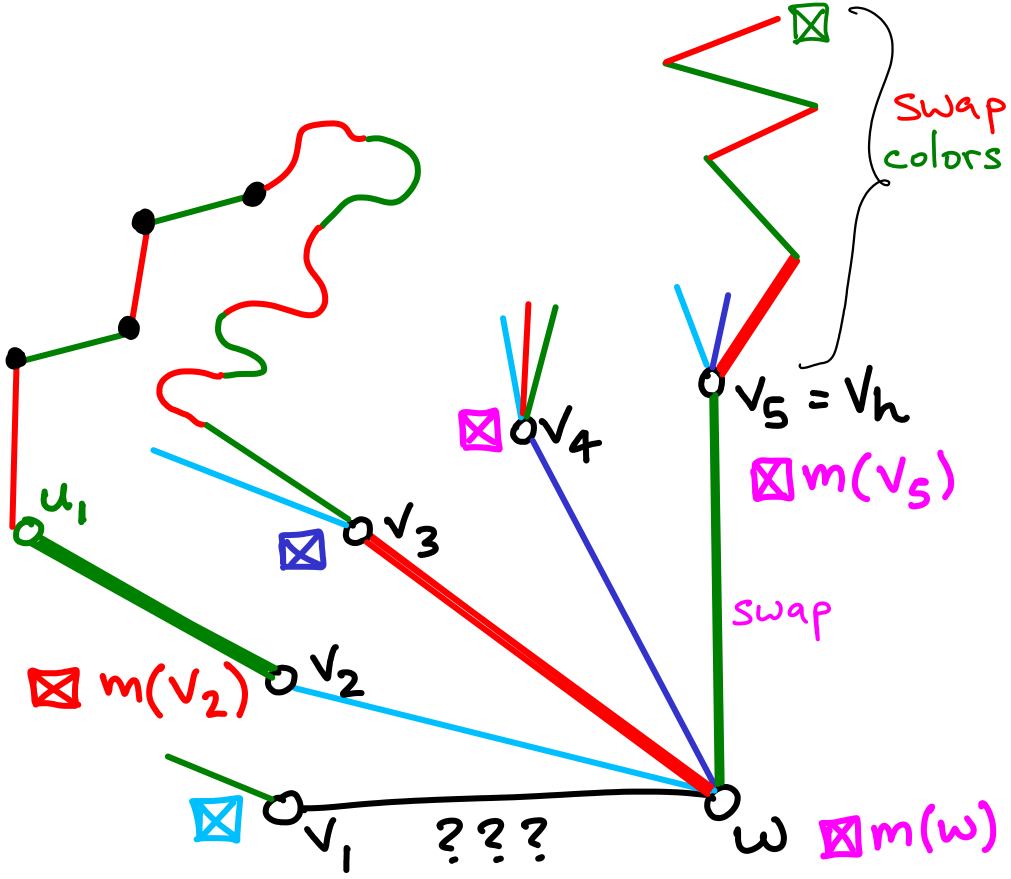
Now look at edge from v_h with color $m(w)$

Extend to a path with same colors as previous path.

↳ must be simple (no cycles) & can't use w at all

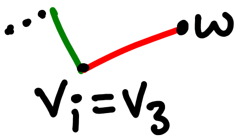
Then swap colors on the v_h path

This starts a chain reaction on the fan



Final case: path ends at w. (it can't go through w because ⊠ m(w))

So path ends w/



Now look at edge from v_h with color m(w)

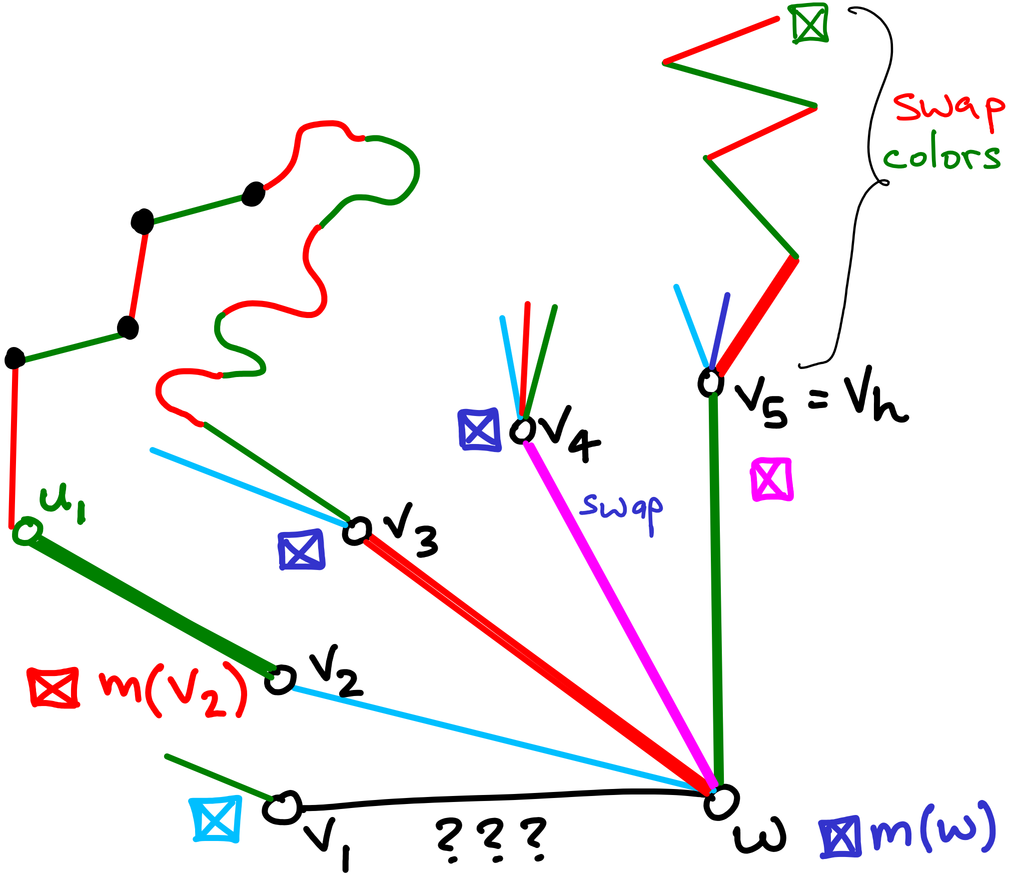
Extend to a path with same colors as previous path.

↳ must be simple (no cycles)

& <u>can't use w at all</u>

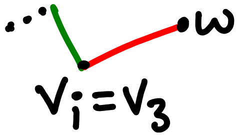
Then swap colors on the v_h path

This starts a chain reaction on the fan



Final case: path ends at w. (it can't go through w because ⊠ m(w))

So path ends w/



Now look at edge from v_h with color m(w)

Extend to a path with same colors as previous path.

↳ must be simple (no cycles) & <u>can't use w at all</u>

Then swap colors on the v_h path

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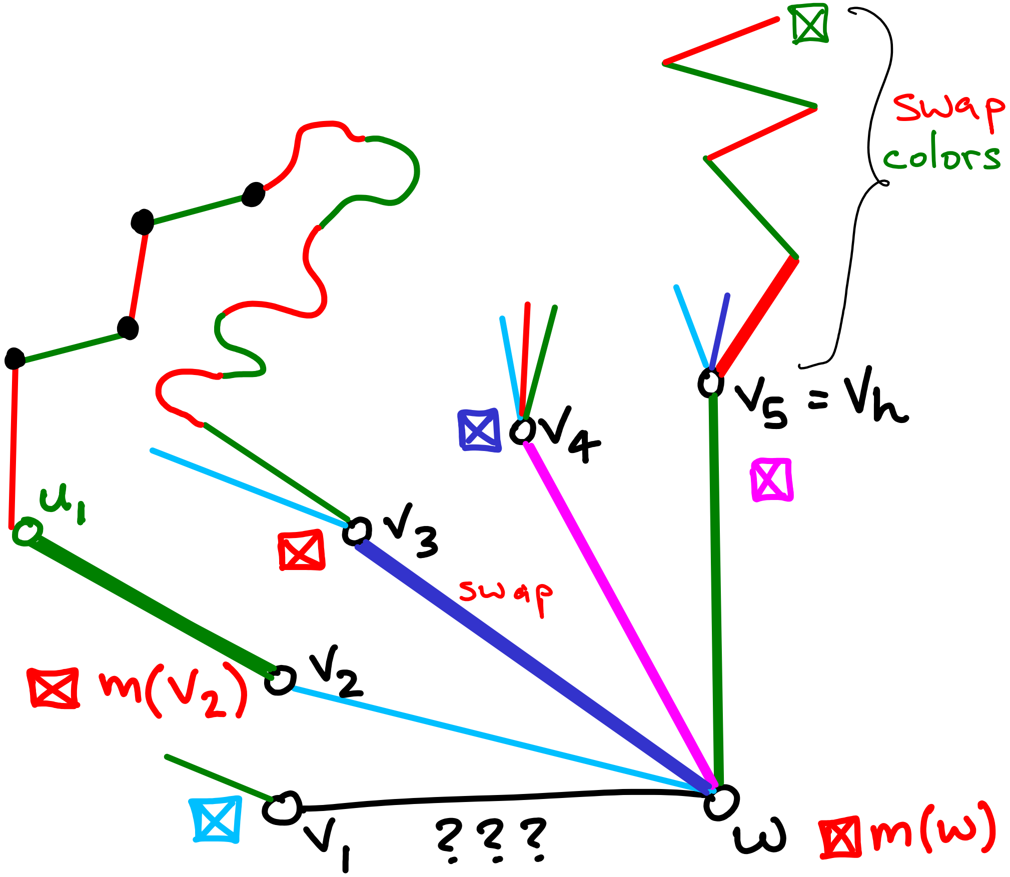
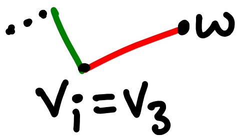


Final case: path ends at w. (it can't go through w because ⊠ m(w))

So path ends w/ 

Now look at edge from v_h with color m(w)

Extend to a path with same colors as previous path.

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Then swap colors on the v_h path

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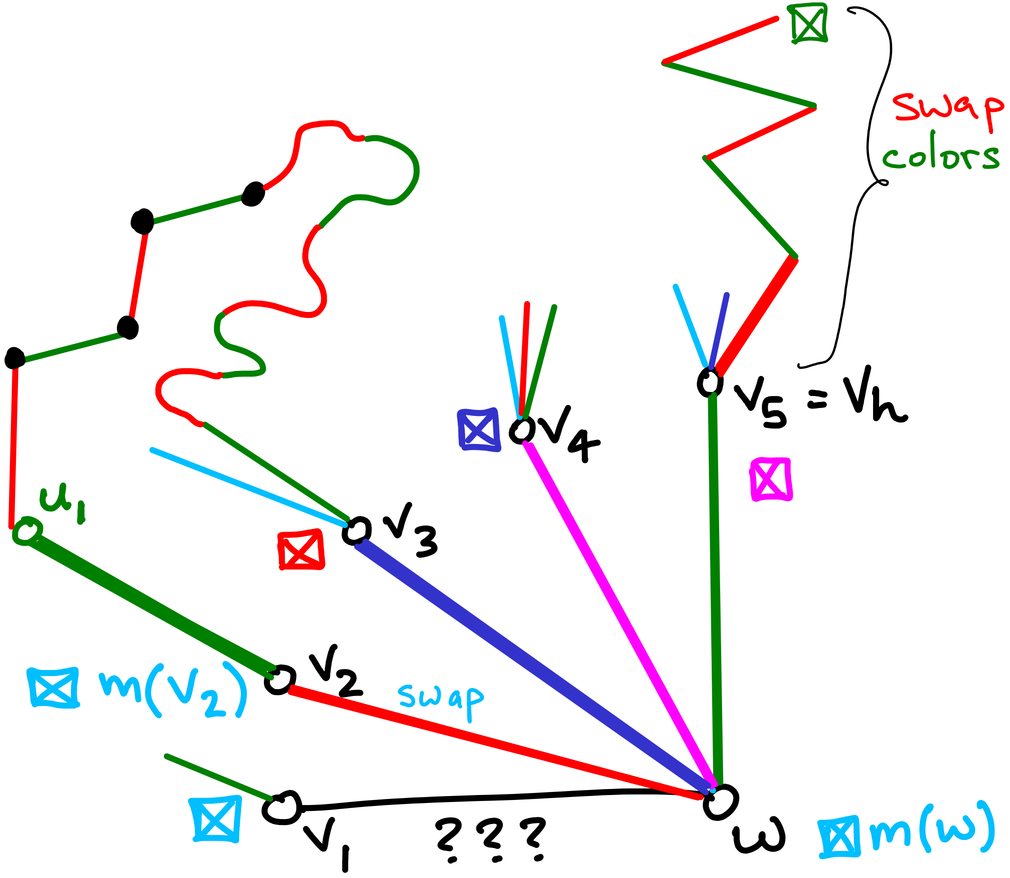
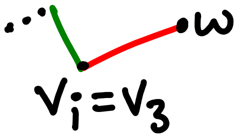


Final case: path ends at w. (it can't go through w because ⊠ $m(w)$)

So path ends w/ 

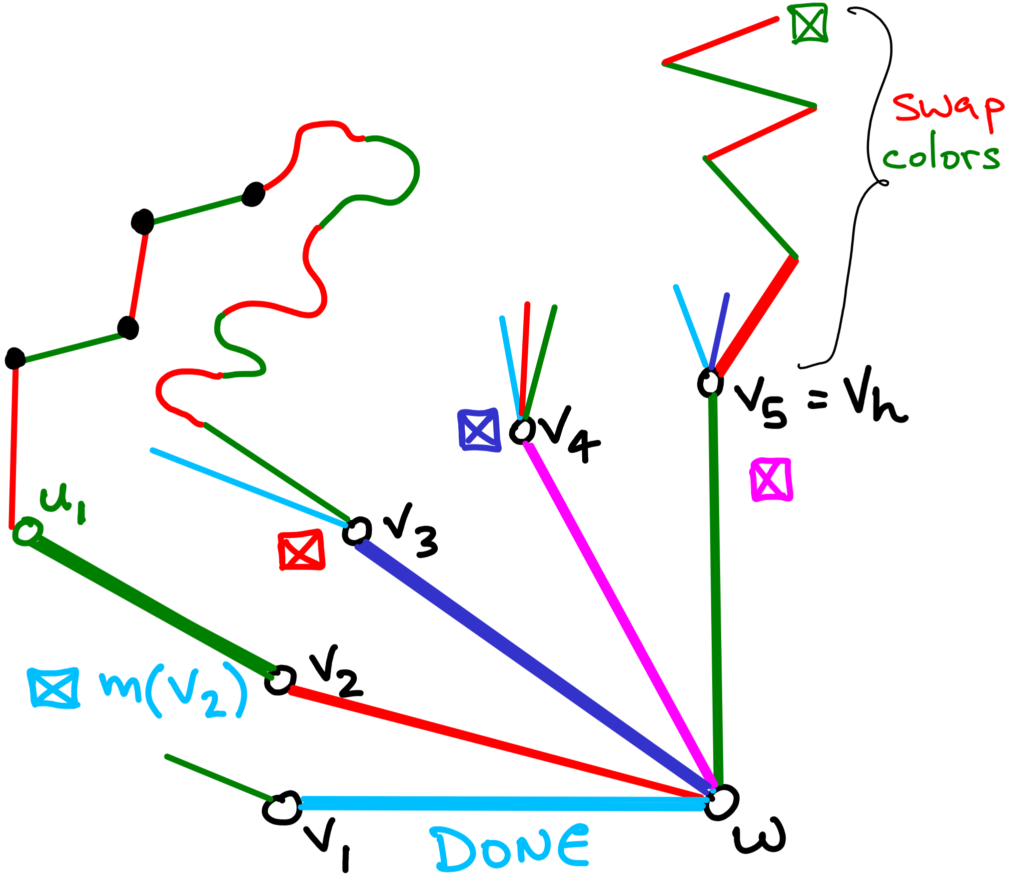
Now look at edge from v_h with color $m(w)$

Extend to a path with same colors as previous path.

↳ must be simple (no cycles) & can't use w at all

Then swap colors on the v_h path

This starts a chain reaction on the fan



Conclusion: every graph w/ max degree Δ
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(there are polynomial algorithms for specific classes of graphs)

$\hookrightarrow$ planar, random, bipartite, etc

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More project topics: algorithms for coloring with $\Delta+1$ colors