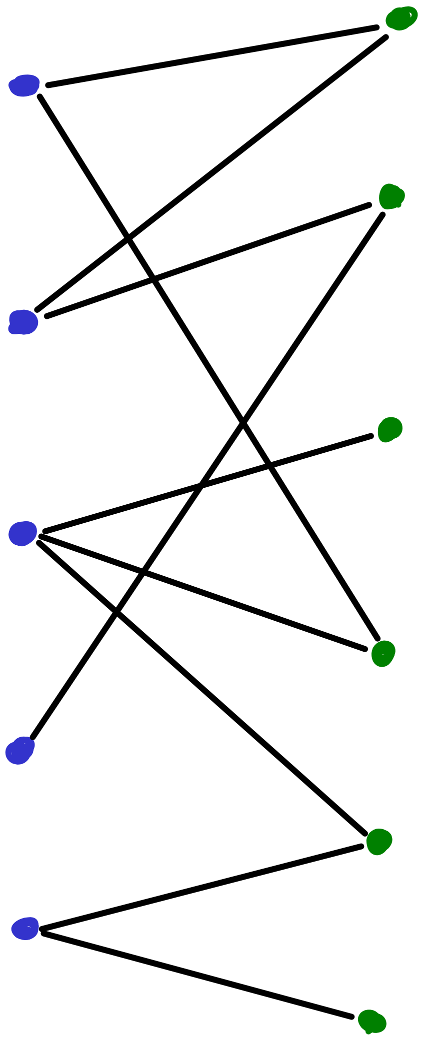
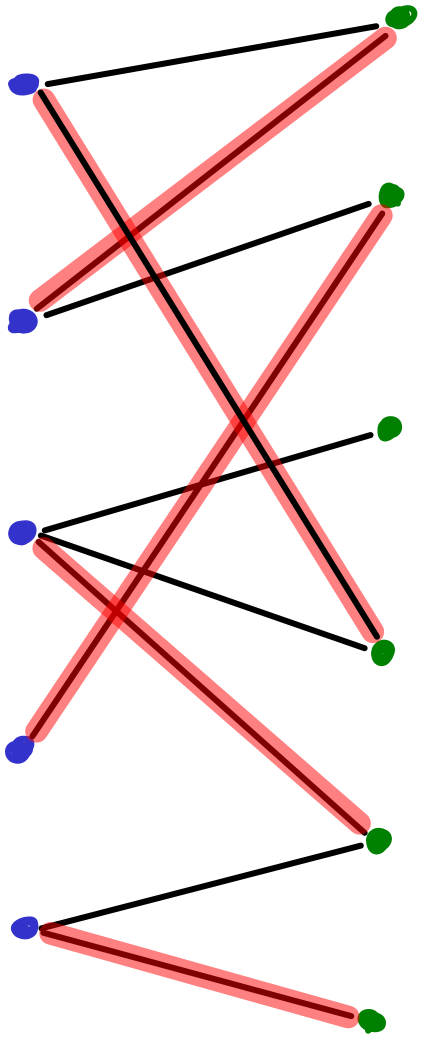
MATCHING in a BIPARTITE GRAPH

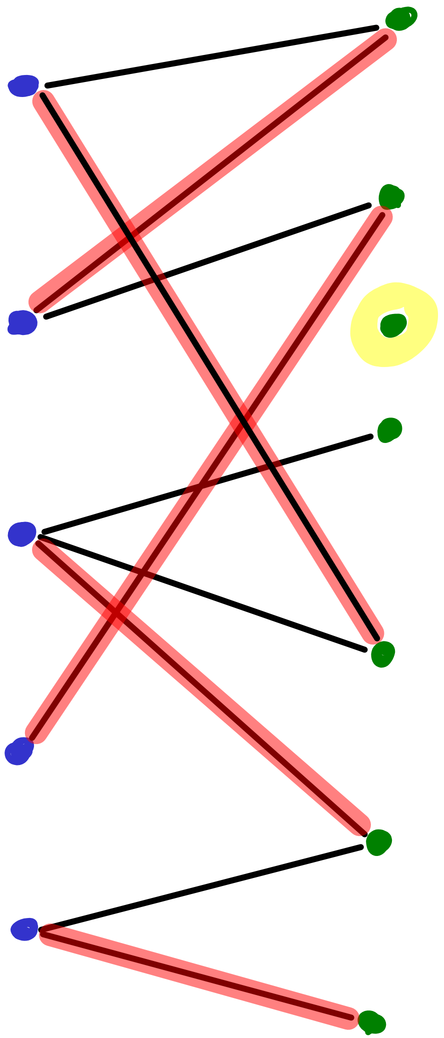


MATCHING in a BIPARTITE GRAPH



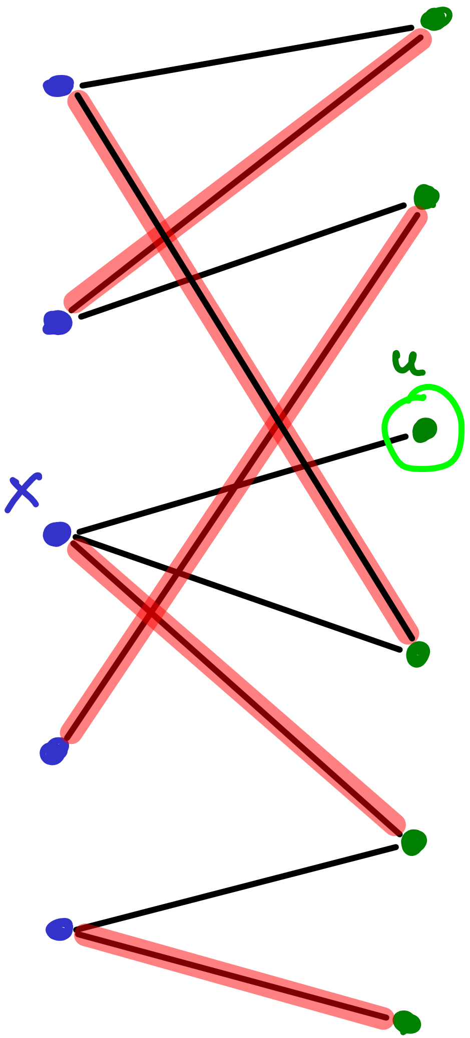
Goal : maximize # independent edges

MATCHING in a BIPARTITE GRAPH



if no incident edges, no hope

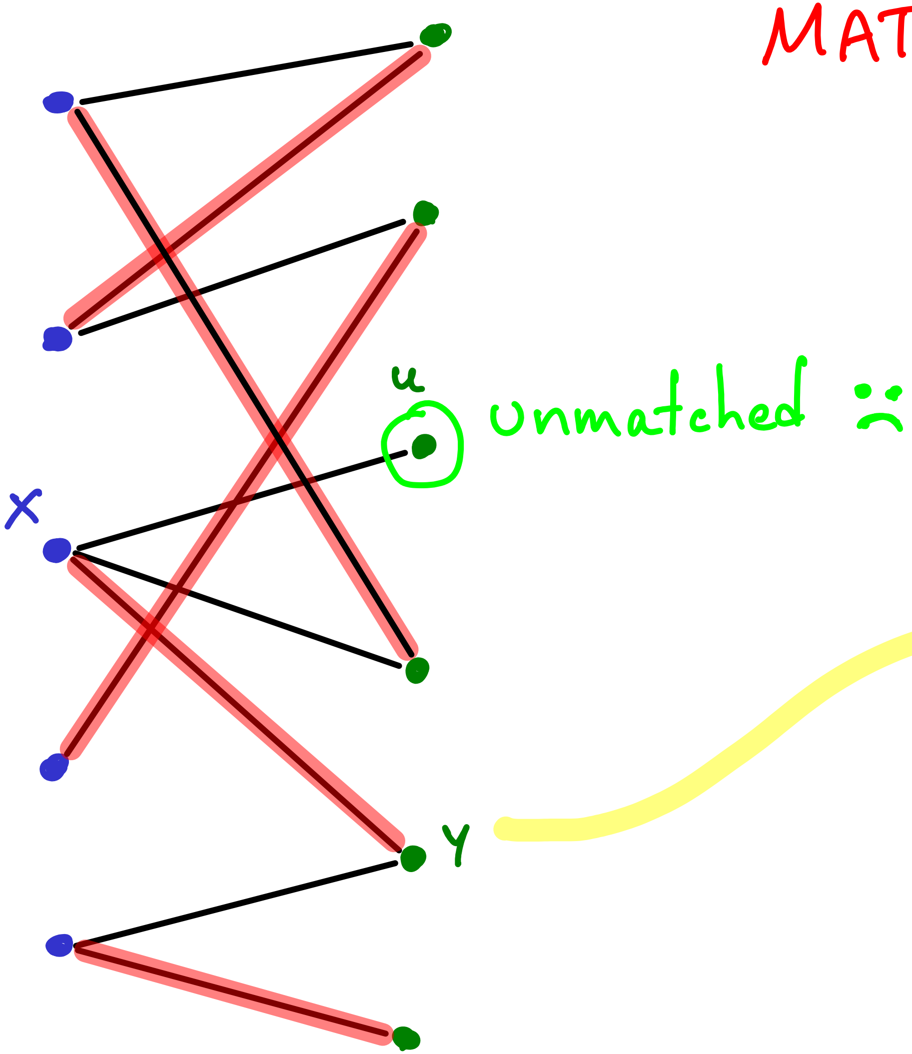
MATCHING in a BIPARTITE GRAPH



unmatched :(

- if no incident edges, no hope
- if $\exists$ edge at u & it is not <u>marked</u>
  then $\exists$ x that is matched elsewhere
  (otherwise match $x \leftrightarrow u$)

MATCHING in a BIPARTITE GRAPH



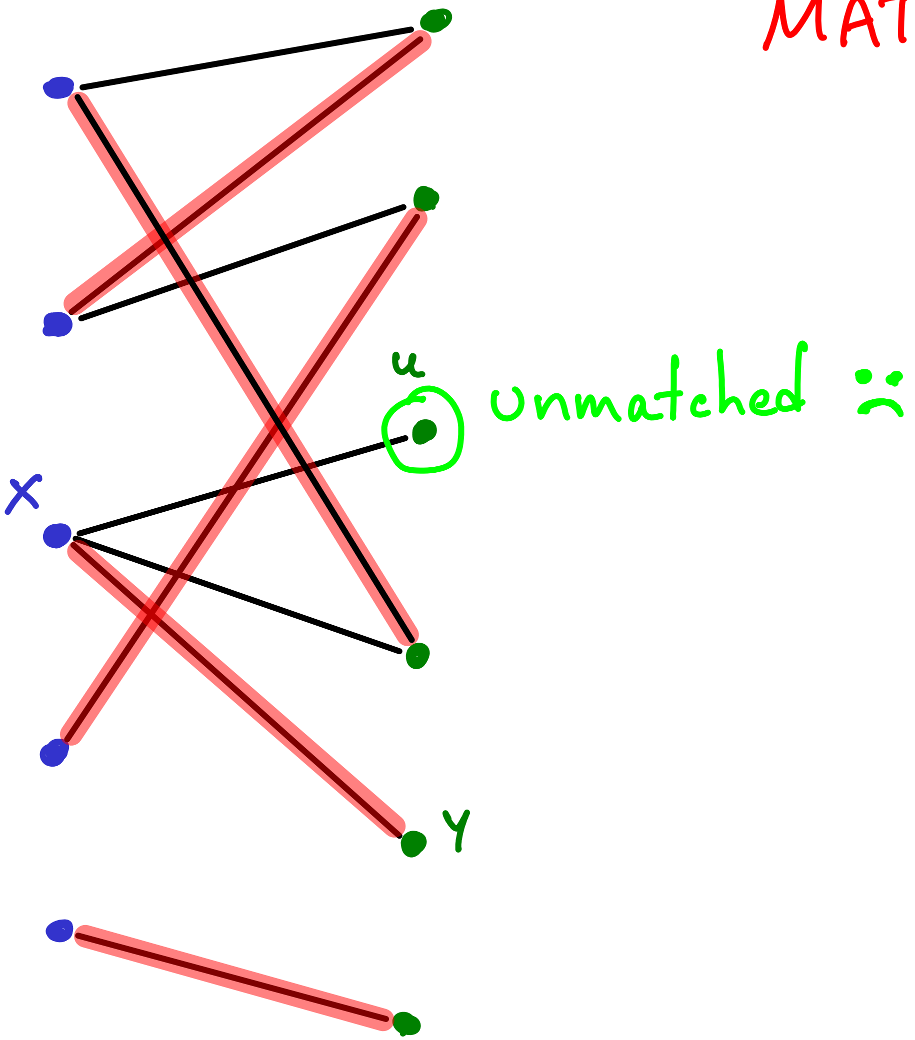
- if no incident edges, no hope
- if $\exists$ edge at u & it is not marked

  then $\exists$ x that is matched elsewhere

  (otherwise match $x \leftrightarrow u$)

so $\exists$ y s.t. $x \leftrightarrow y$

MATCHING in a BIPARTITE GRAPH



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- if $\exists$ edge at u & it is not marked

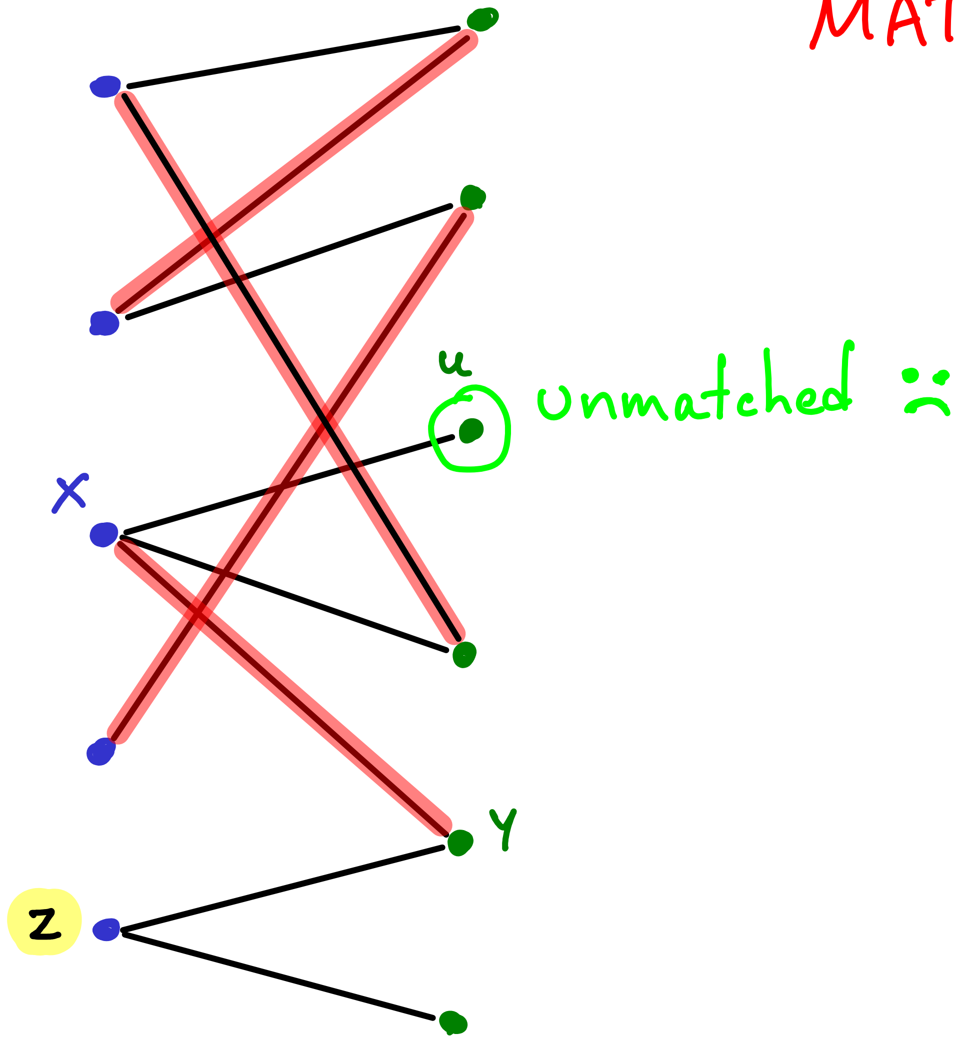
  then $\exists$ x that is matched elsewhere.

  (otherwise match $x \leftrightarrow u$)

  so $\exists$ y s.t. $x \leftrightarrow y$

  Either y has no other edges (story ends)

MATCHING in a BIPARTITE GRAPH



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- if $\exists$ edge at u & it is not marked

  then $\exists$ x that is matched elsewhere

  (otherwise match $x \leftrightarrow u$)

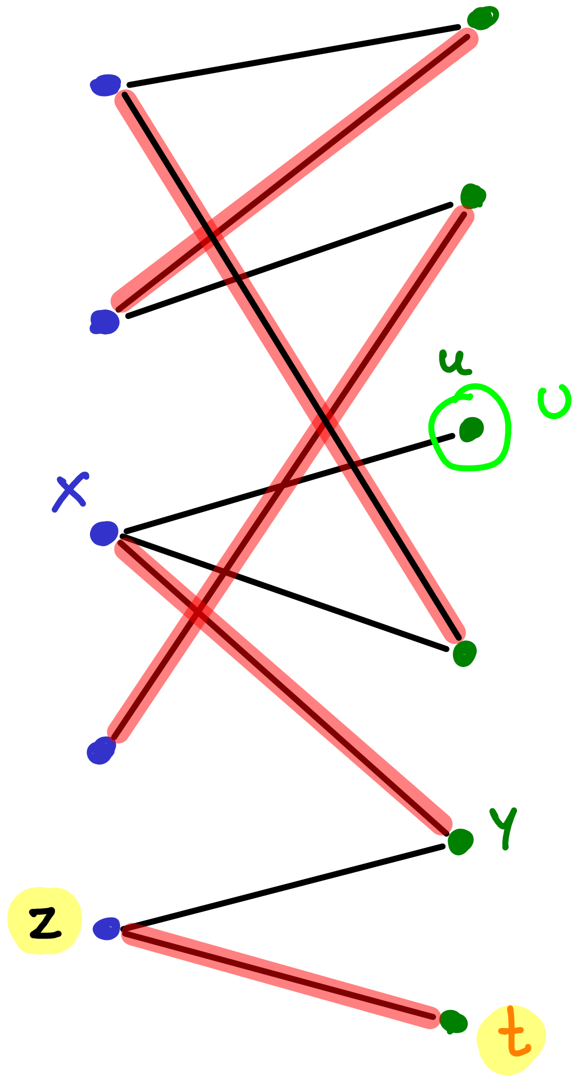
  So $\exists$ y s.t. $x \leftrightarrow y$

  Either y has no other edges (story ends)

  OR $\exists$ z

  If z is not matched then improve matching!

MATCHING in a BIPARTITE GRAPH



unmatched :(

- if no incident edges, no hope
- if ∃ edge at u & it is not marked

  then ∃ x that is matched elsewhere.

  (otherwise match x ⟷ u)

  So ∃ y s.t. x ⟷ y

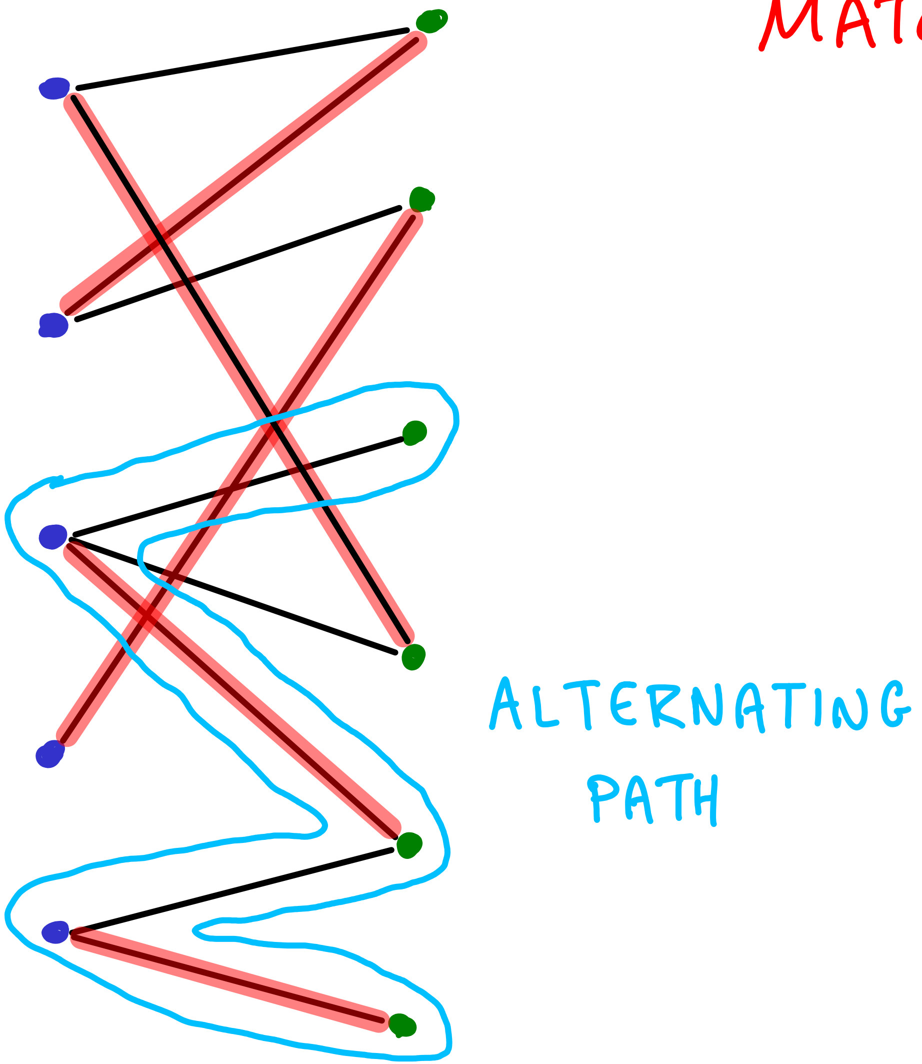
  Either y has no other edges (story ends)

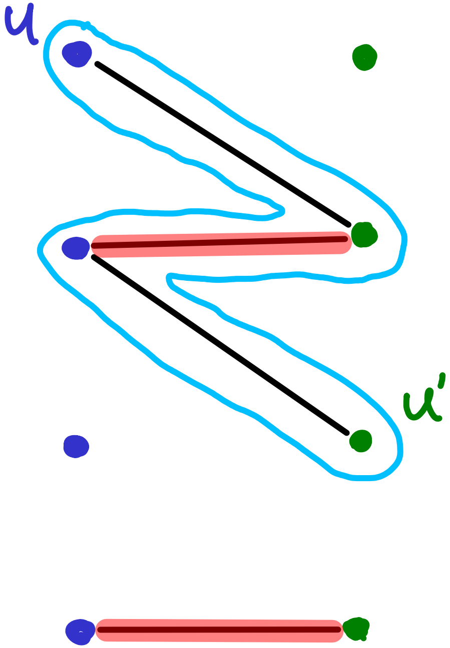
  OR ∃ z

  If z is not matched then improve matching!

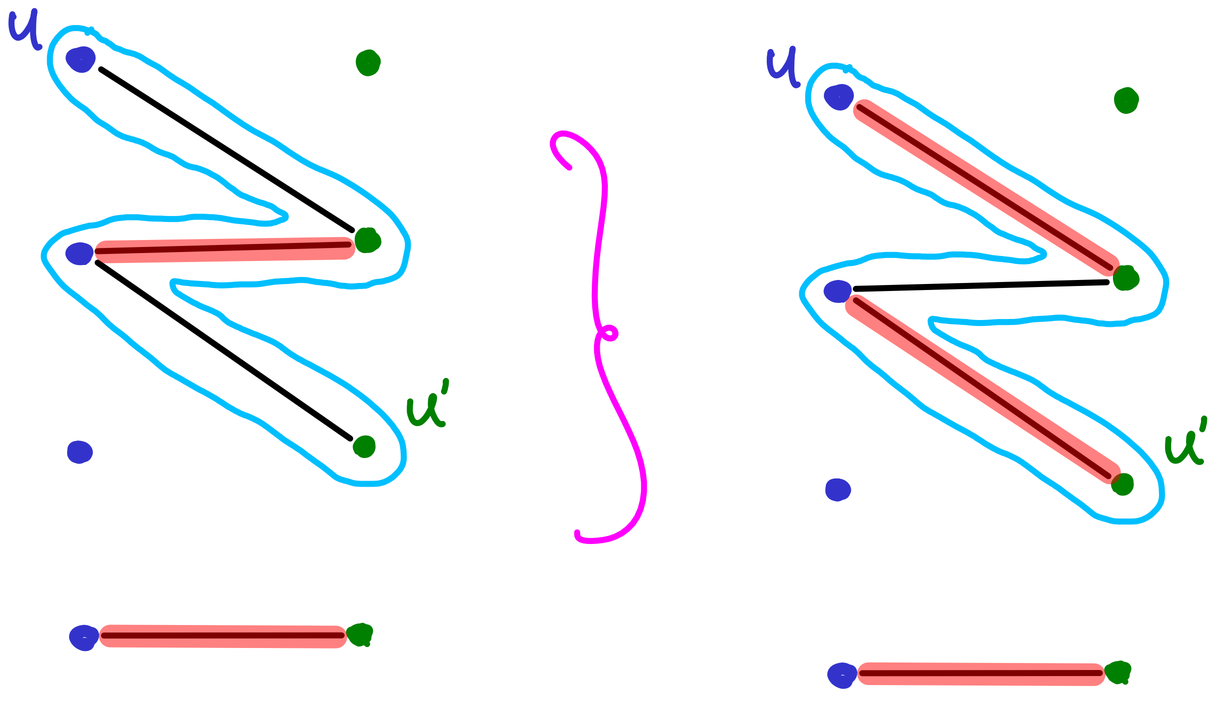
  else ∃ z ⟷ t   etc

MATCHING in a BIPARTITE GRAPH



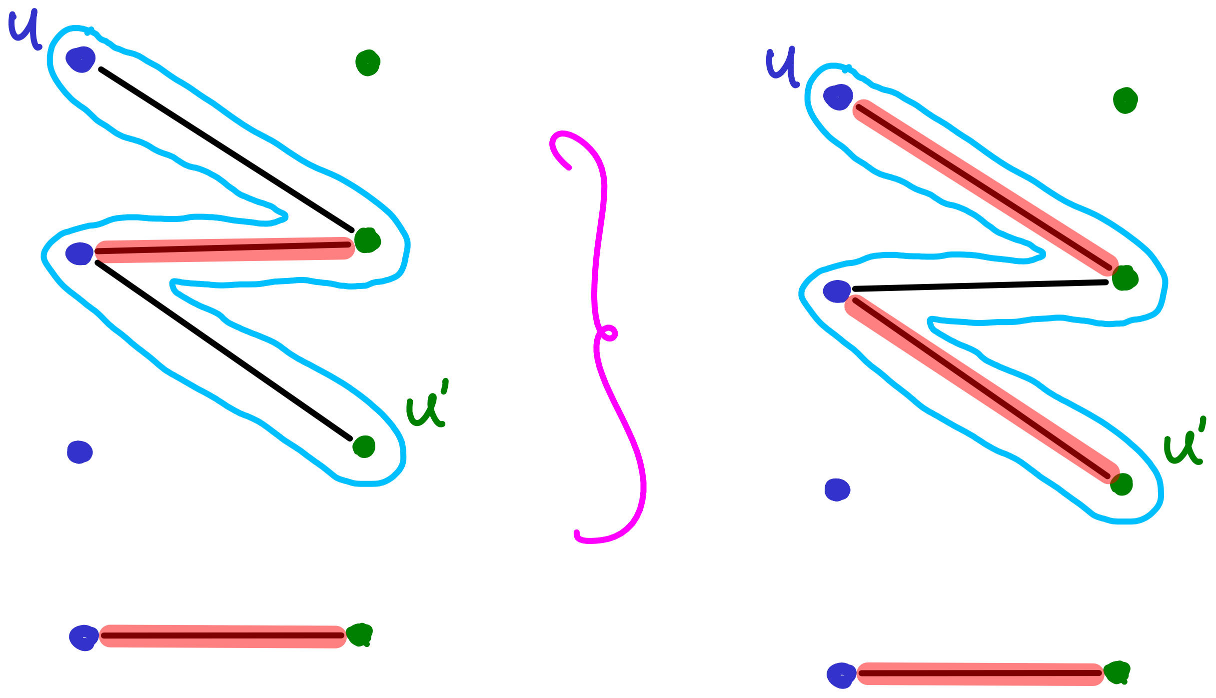


AUGMENTING PATH

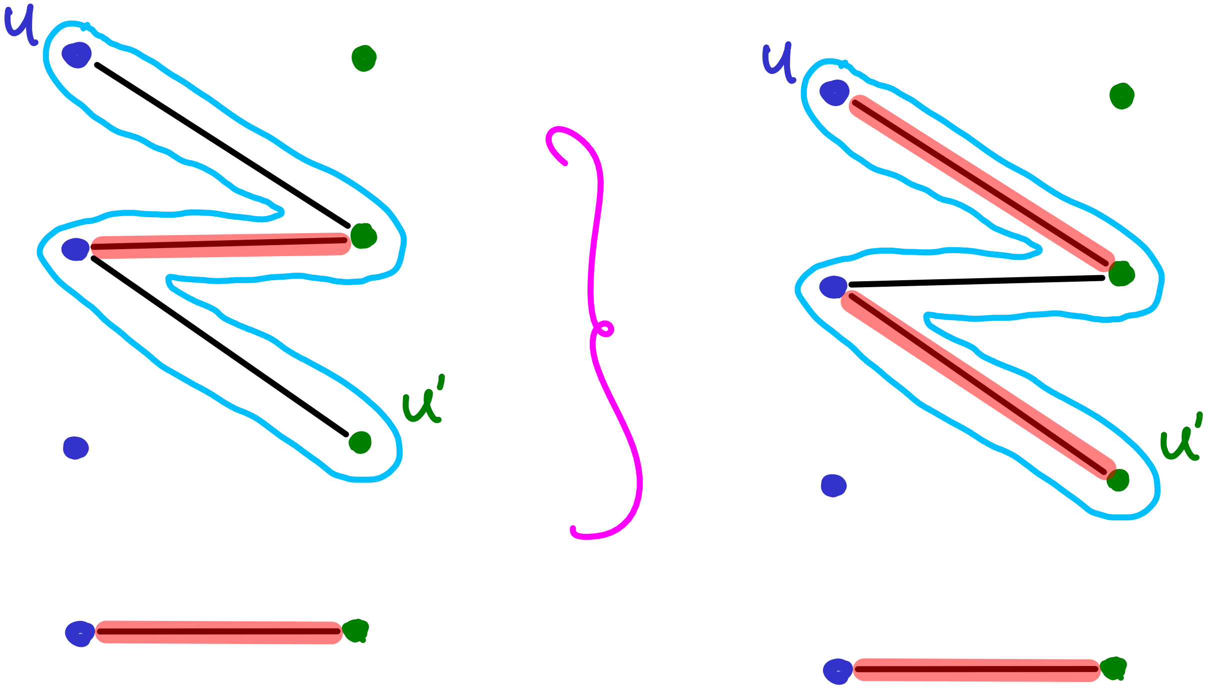


AUGMENTING PATH → Improve matching

Is a matching optimal if no augmenting path exists?



AUGMENTING PATH $\longrightarrow$ Improve matching



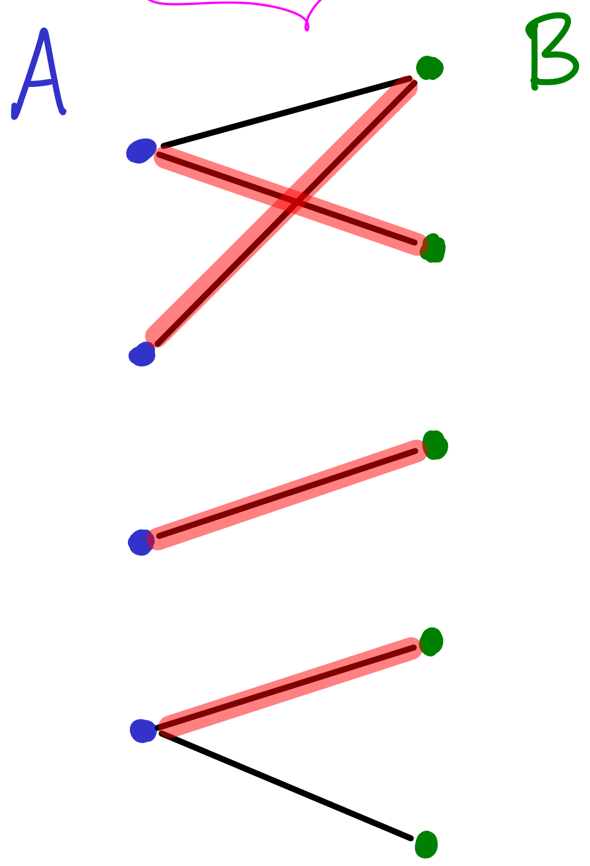
Is a matching optimal if no augmenting path exists?

Algorithm & time complexity to find an avg. path ? ... or an optimal matching?

AUGMENTING PATH → Improve matching

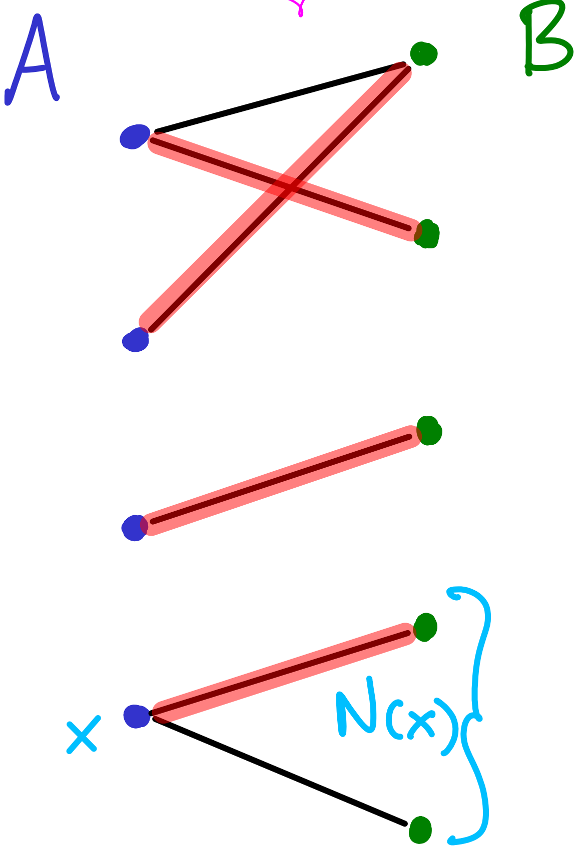
TBD

All vertices of A matched



All vertices of A matched → necessary

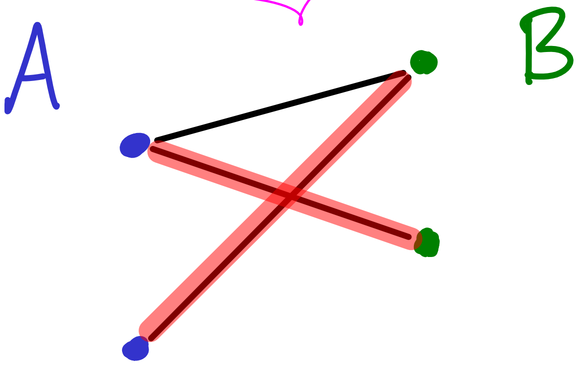
↳ every vertex has a neighbor

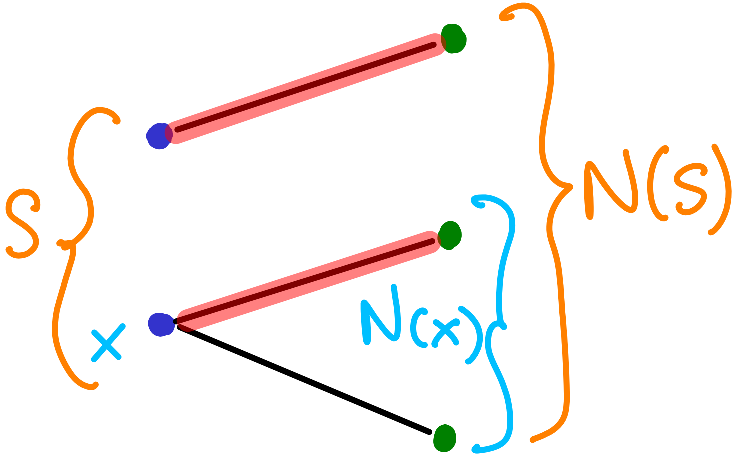


All vertices of A matched $\rightarrow$ <u>necessary</u>

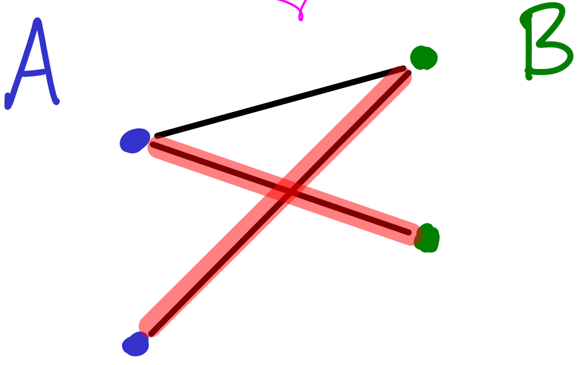
↳ every vertex has a neighbor

↳ every group of S vertices in A has $\geq |S|$ neighbors.





All vertices of A matched → necessary



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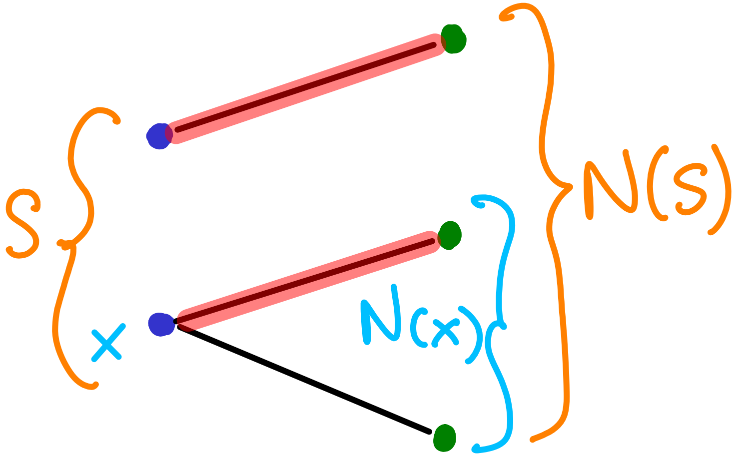


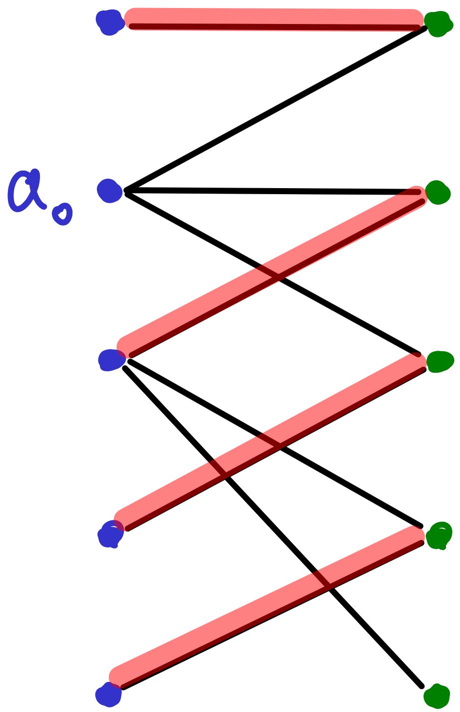
Also sufficient: (Hall's theorem)

All vertices in A will be matched if for every $S \subseteq A$

$$|N(S)| \geq |S|$$

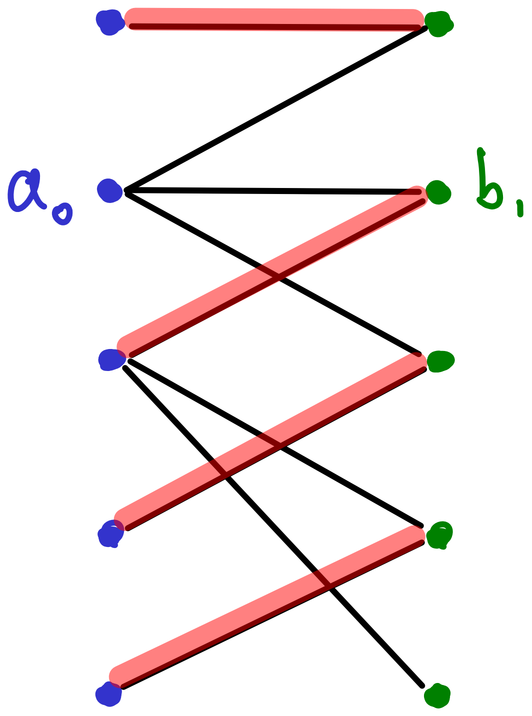
...

Start w/ best matching. Suppose $\underbrace{|N(S)| \geq |S|}_{\text{for all } S \subseteq A}$ but a_0 unmatched



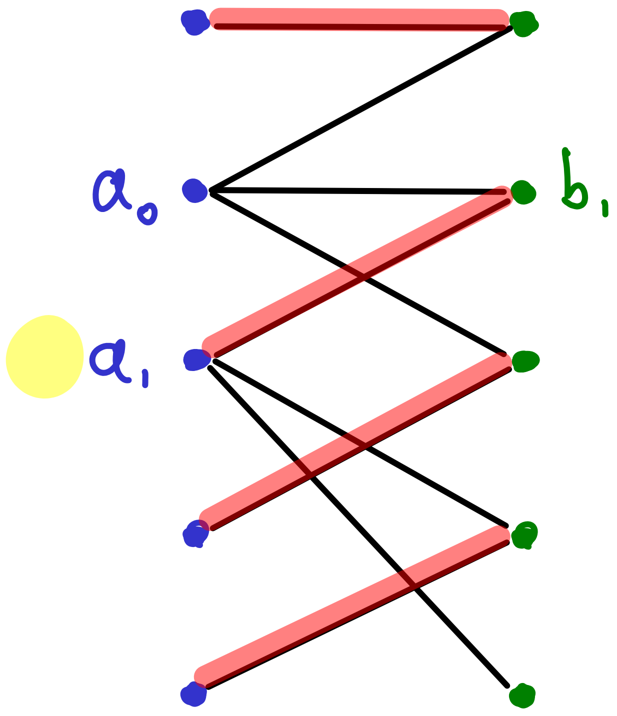
proof of Hall's theorem, by contradiction

Start w/ best matching. Suppose $|N(S)| \geq |S|$ but a_0 unmatched



$\exists\ b_1$ adjacent to a_0 $\quad (|N(a_0)| \geq 1)$

Start w/ best matching. Suppose $|N(S)| \geq |S|$ but a_0 unmatched

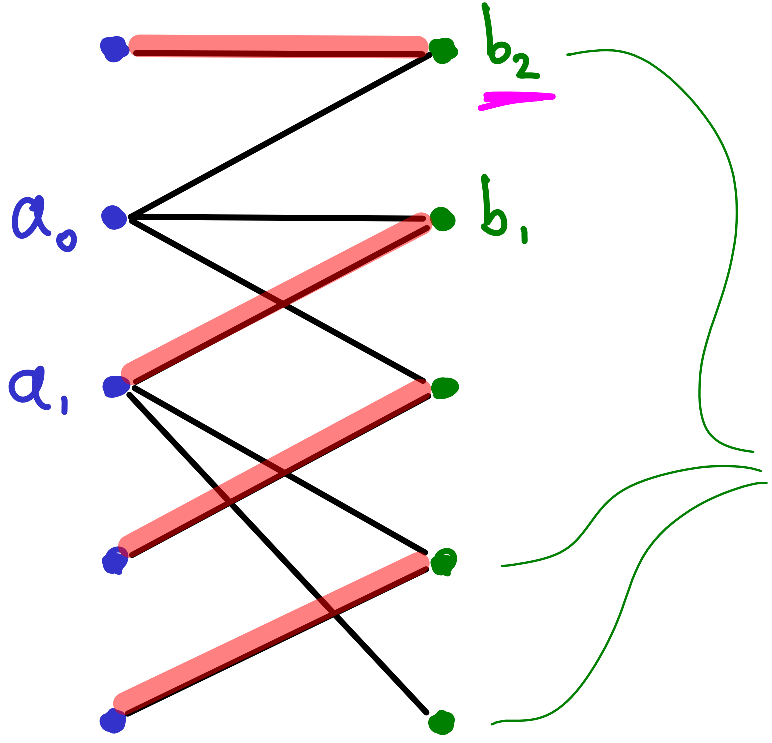


$\exists$ b_1 adjacent to a_0 $(|N(a_0)| \geq 1)$

b_1 matches to some a_1

(otherwise match a_0 to b_1)

Start w/ best matching.



Suppose $|N(S)| \geq |S|$ but a_0 unmatched

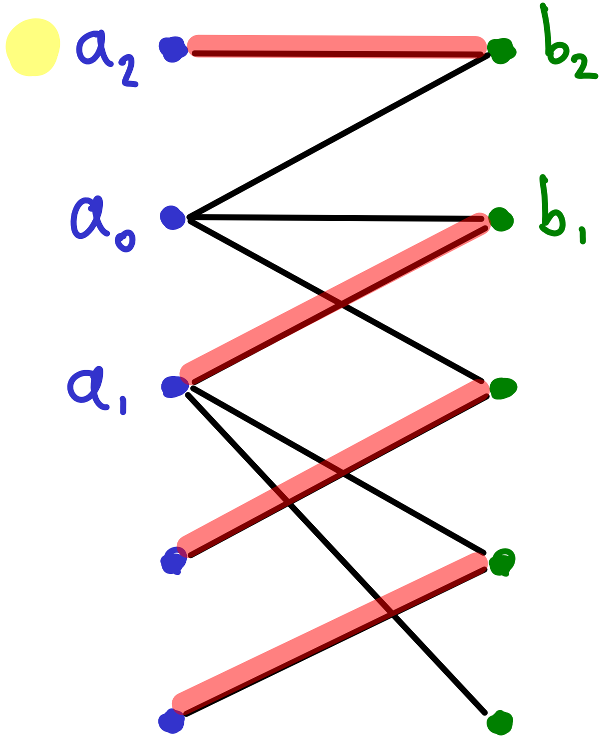
$\exists\ b_1$ adjacent to a_0 $\quad (|N(a_0)| \geq 1)$

b_1 matches to some a_1

(otherwise match a_0 to b_1)

Next, if $\exists$ other vertex adjacent to a_0 or a_1
label it b_2

Start w/ best matching. Suppose $|N(S)| \geq |S|$ but a_0 unmatched



$\exists\ b_1$ adjacent to a_0 $\quad (|N(a_0)| \geq 1)$

b_1 matches to some a_1

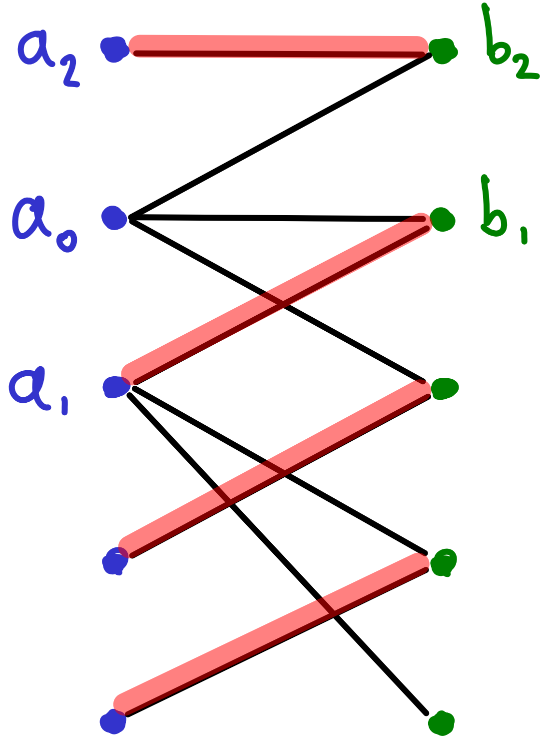
(otherwise match a_0 to b_1)

Next, if $\exists$ other vertex adjacent to a_0 or a_1, label it b_2

if b_2 matches to something, label it a_2

can't be a_0, a_1

Start w/ best matching.



$a_0 b_1 a_1 b_2 a_2 \ldots$

Suppose $|N(S)| \geq |S|$ but a_0 unmatched

$\exists\ b_1$ adjacent to a_0 $\quad (|N(a_0)| \geq 1)$

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Next, if $\exists$ other vertex adjacent to a_0 or a_1
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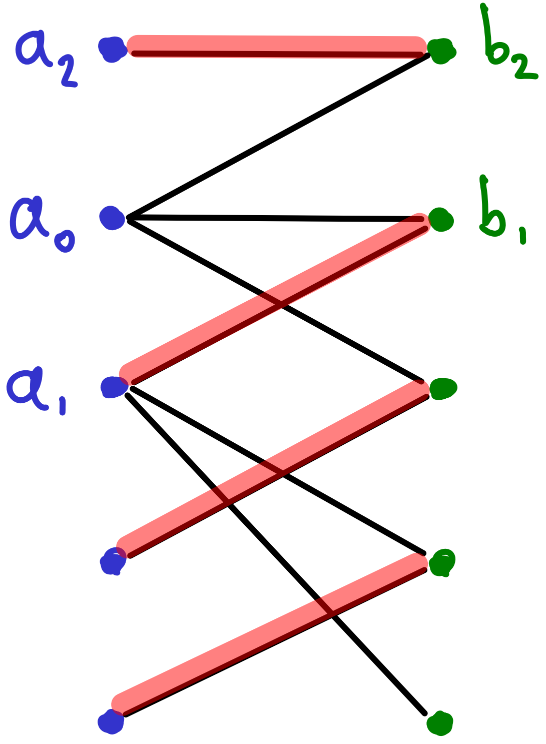
While possible, extend this alternating sequence:

1) Add/label a_i if it matches to b_i

Start w/ best matching.



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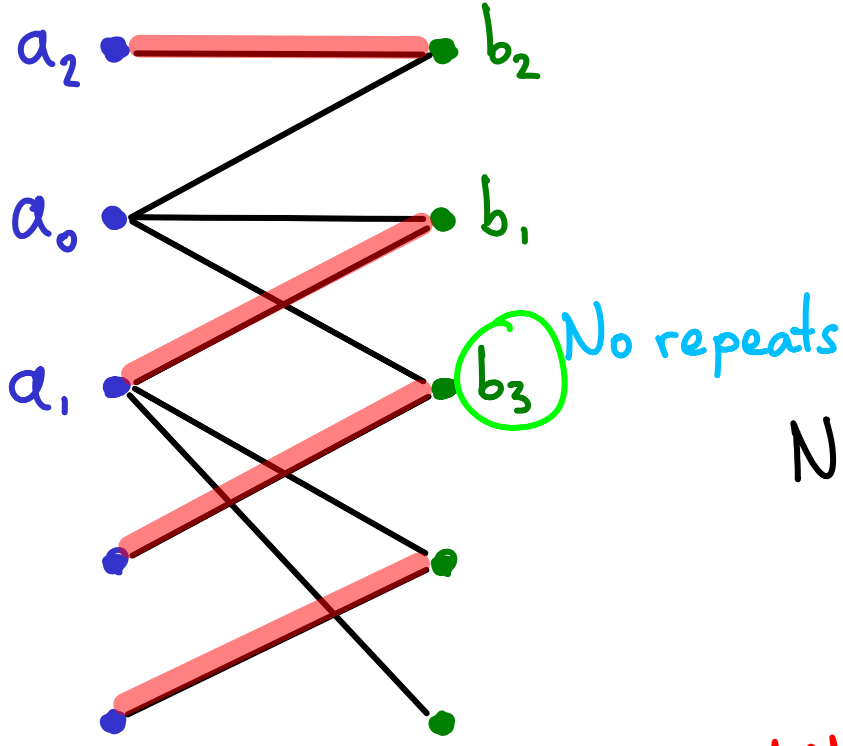
2) Add b_i if it is in $N(a_0 \ldots a_{i-1})$

Start w/ best matching.



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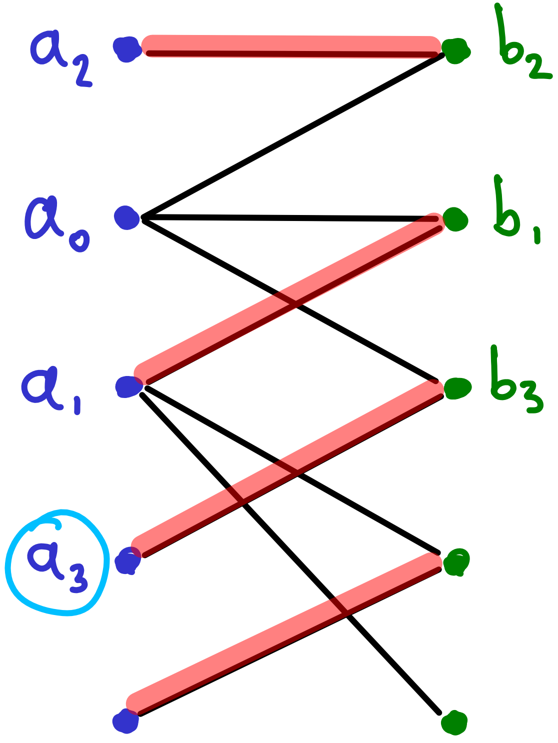

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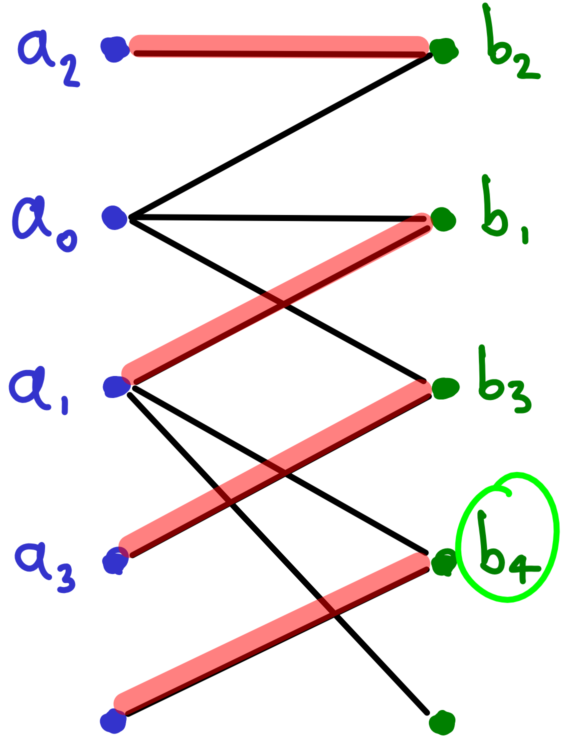

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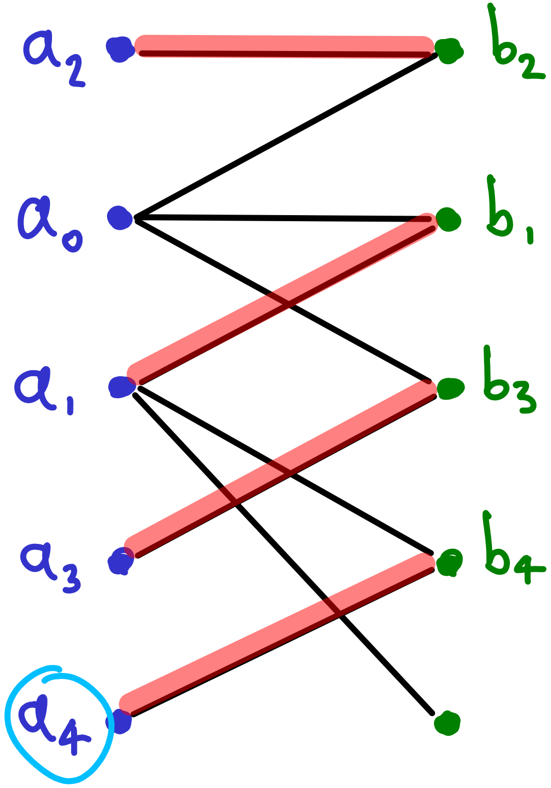

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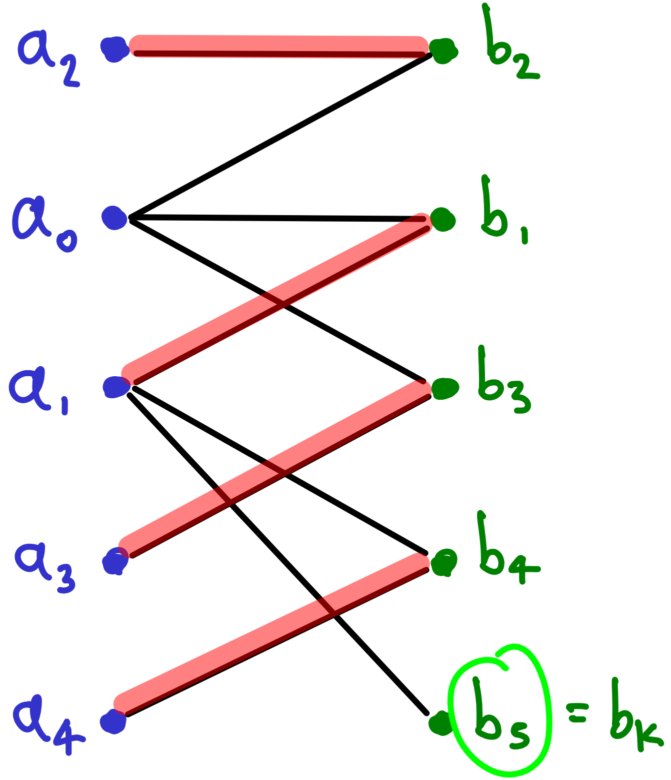

$a_0 b_1 a_1 b_2 a_2 \dots b_k$

Suppose $|N(S)| \geq |S|$ but a_0 unmatched

$\exists\, b_1$ adjacent to a_0 $\quad (|N(a_0)| \geq 1)$

b_1 matches to some a_1

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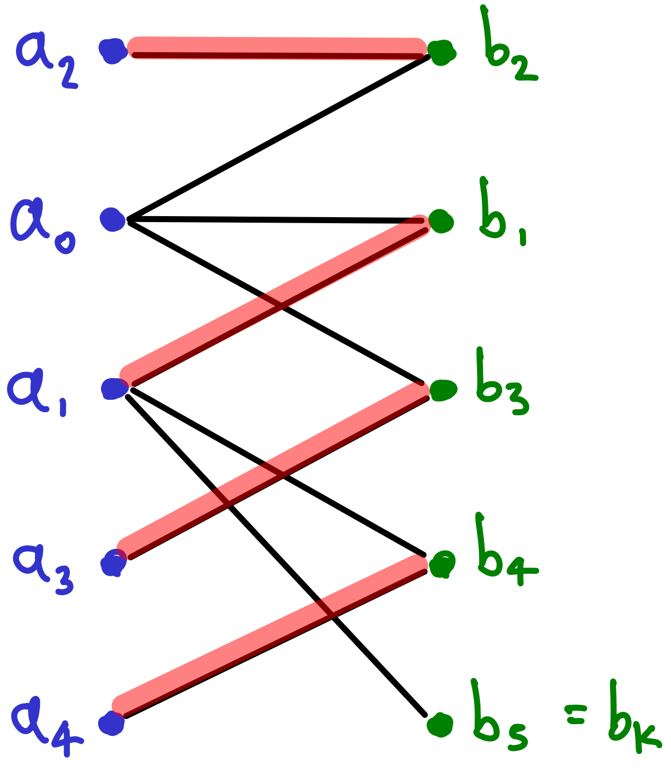
Add/label a_i if it matches to b_i

Add b_i if it is in $N(a_0 \dots a_{i-1})$

Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



$a_0 b_1 a_1 b_2 a_2 \dots b_K$

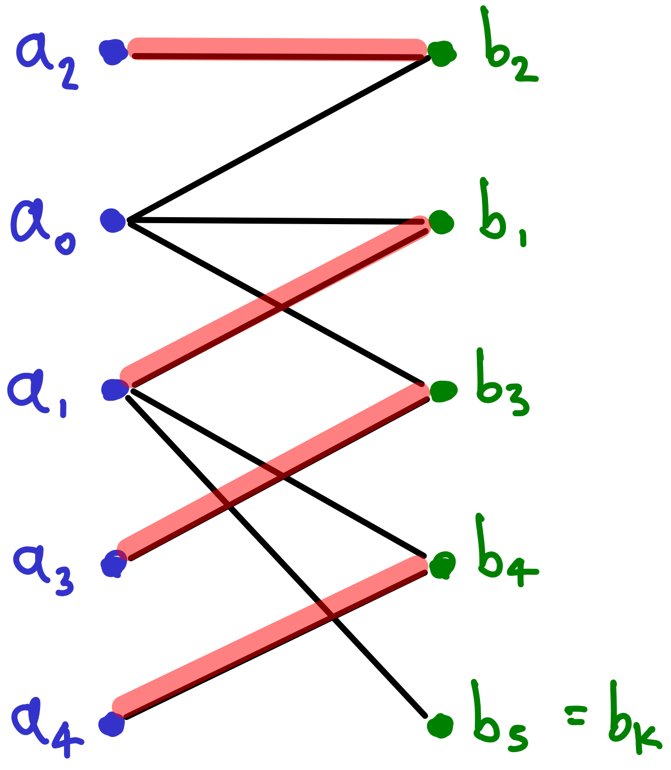
While possible, extend this alternating sequence:

Add/label a_i if it matches to b_i

Add b_i if it is in $N(a_0 \dots a_{i-1})$

Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



While possible, extend this alternating sequence:

Add/label a_i if it matches to b_i

Add b_i if it is in $N(a_0 \ldots a_{i-1})$

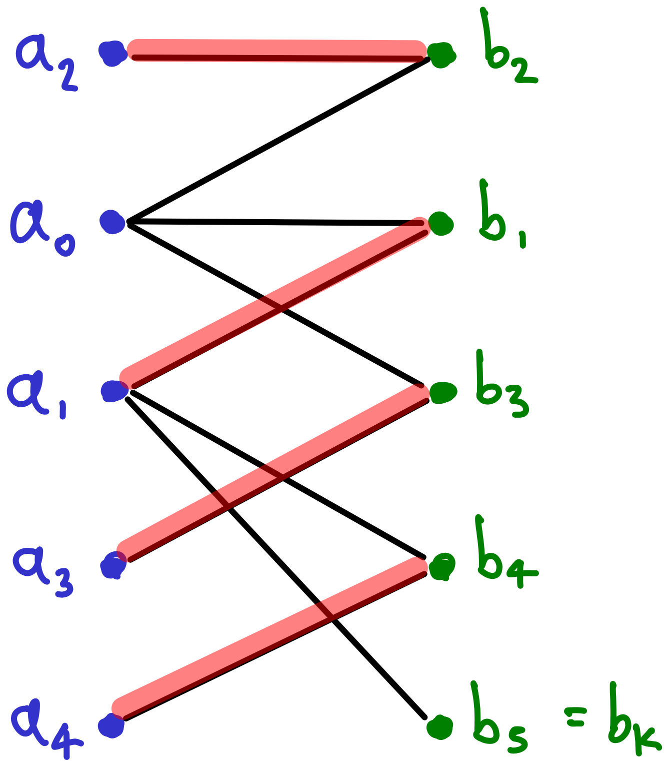
Can this end in A at some a_k?

$a_0 b_1 a_1 b_2 a_2 \ldots b_k$

or

$a_0 b_1 a_1 b_2 a_2 \ldots a_k$?

Start w/ best matching.



$a_0 b_1 a_1 b_2 a_2 \dots b_k$

or

$a_0 b_1 a_1 b_2 a_2 \dots a_k$?

Suppose $|N(S)| \geq |S|$ but a_0 unmatched

While possible, extend this alternating sequence:

Add/label a_i if it matches to b_i

Add b_i if it is in $N(a_0 \dots a_{i-1})$

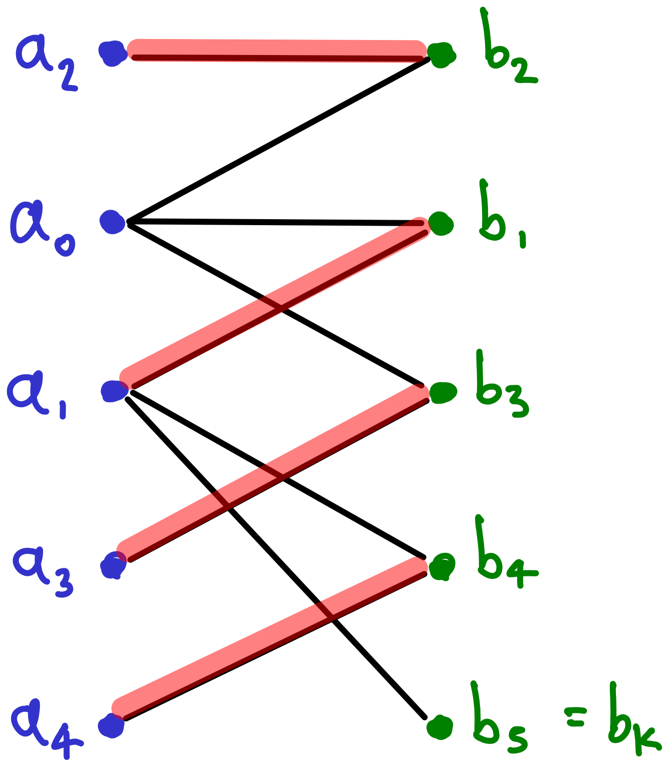
Can this end in A at some a_k ?

No because $|N(a_0 \dots a_k)| \geq k+1$

& we've only used $b_1 \dots b_k$

$\exists$ some other $b \neq b_1 \dots b_k$ in $N(a_0 \dots a_k)$

Start w/ best matching.



$a_0 b_1 a_1 b_2 a_2 \ldots b_k$

or

~~$a_0 b_1 a_1 b_2 a_2 \ldots a_k$?~~

Suppose $|N(S)| \geq |S|$ but a_0 unmatched

While possible, extend this alternating sequence:

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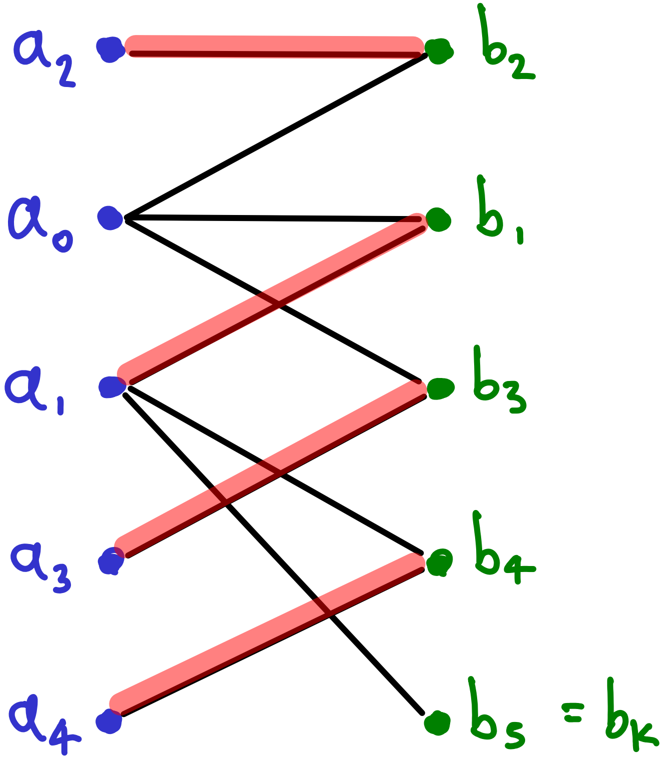
& we've only used $b_1 \ldots b_k$

$\exists$ some other $b \neq b_1 \ldots b_k$ in $N(a_0 \ldots a_k)$

Then ... ?

Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



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$\exists$ some other $b \neq b_1 \ldots b_k$ in $N(a_0 \ldots a_k)$

$a_0 b_1 a_1 b_2 a_2 \ldots b_k$

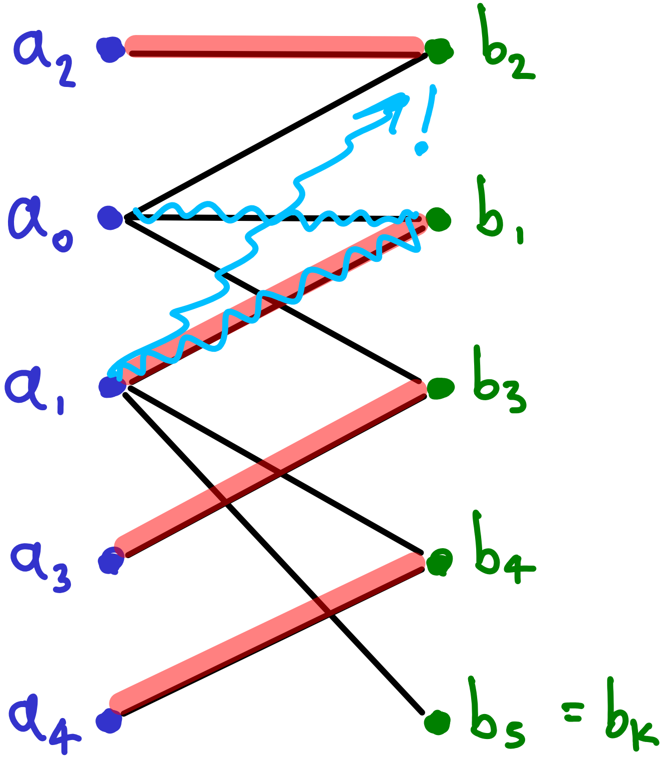
or

~~$a_0 b_1 a_1 b_2 a_2 \ldots a_k$?~~

Is our alternating sequence an alternating path?

Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



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Add/label a_i if it matches to b_i

Add b_i if it is in $N(a_0 \ldots a_{i-1})$

Can this end in A at some a_k?

No because $|N(a_0 \ldots a_k)| \geq k+1$

& we've only used $b_1 \ldots b_k$

$\exists$ some other $b \neq b_1 \ldots b_k$ in $N(a_0 \ldots a_k)$

$a_0 b_1 a_1 b_2 a_2 \ldots b_k$

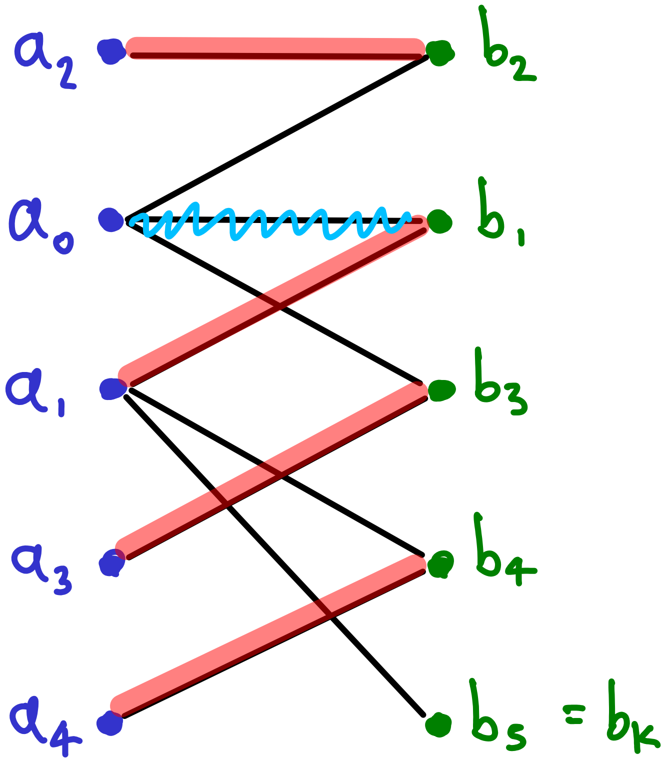
or

$\cancel{a_0 b_1 a_1 b_2 a_2 \ldots a_k}$?

Not an alternating path

Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



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$\exists$ some other $b \neq b_1 \dots b_k$ in $N(a_0 \dots a_k)$

$a_0 b_1 a_1 b_2 a_2 \dots b_k$

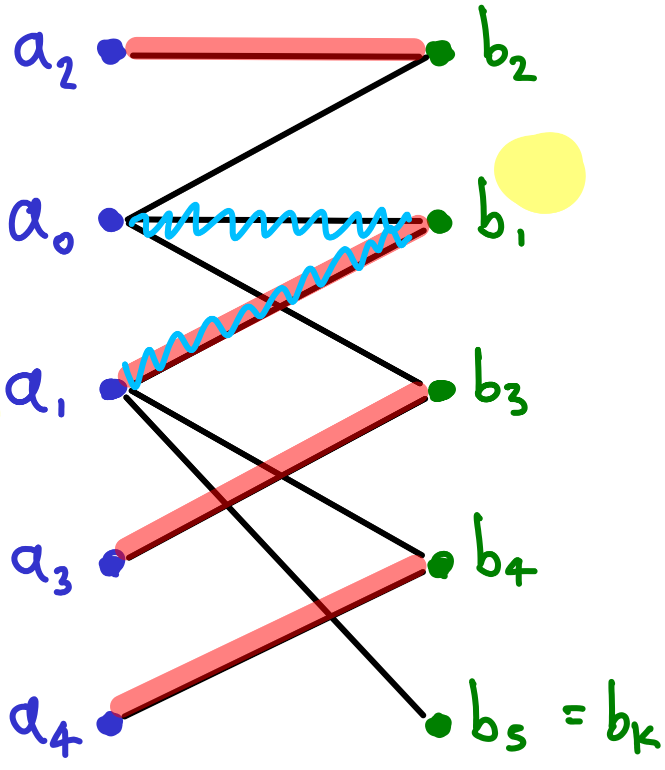
or

$\cancel{a_0 b_1 a_1 b_2 a_2 \dots a_k\ ?}$

We know there is no match $a_0 b_1$

Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



While possible, extend this alternating sequence:

Add/label a_i if it matches to b_i

Add b_i if it is in $N(a_0 \ldots a_{i-1})$

Can this end in A at some a_k ?

No because $|N(a_0 \ldots a_k)| \geq k+1$

& we've only used $b_1 \ldots b_k$

$\exists$ some other $b \neq b_1 \ldots b_k$ in $N(a_0 \ldots a_k)$

We know b_1 matches a_1

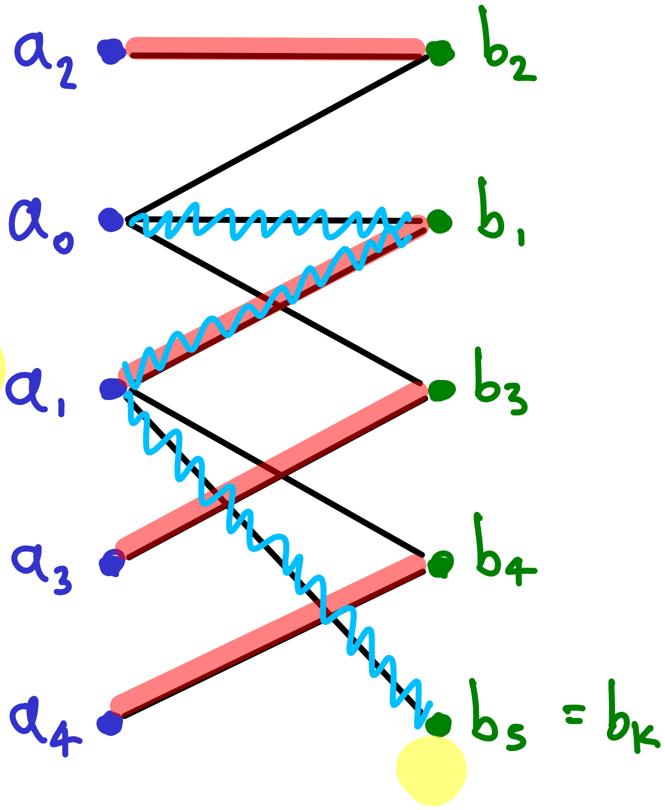
$a_0 b_1 a_1 b_2 a_2 \ldots b_k$

or

~~$a_0 b_1 a_1 b_2 a_2 \ldots a_k$?~~

Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



While possible, extend this alternating sequence:

Add/label a_i if it matches to b_i

Add b_i if it is in $N(a_0 \ldots a_{i-1})$

$a_0 b_1 a_1 b_2 a_2 \ldots b_K$

or

~~$a_0 b_1 a_1 b_2 a_2 \ldots a_K$?~~

Can this end in A at some a_K ?

No because $|N(a_0 \ldots a_K)| \geq K+1$

& we've only used $b_1 \ldots b_K$

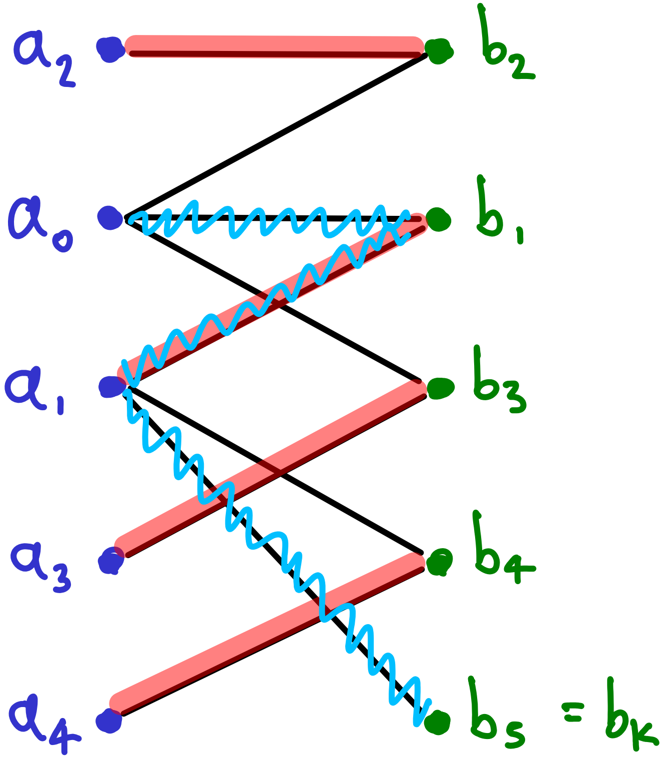
$\exists$ some other $b \neq b_1 \ldots b_K$ in $N(a_0 \ldots a_K)$

a_1 must have a non-match with $b \ldots$ etc

(edge)

Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



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No because $|N(a_0 \ldots a_k)| \geq k+1$

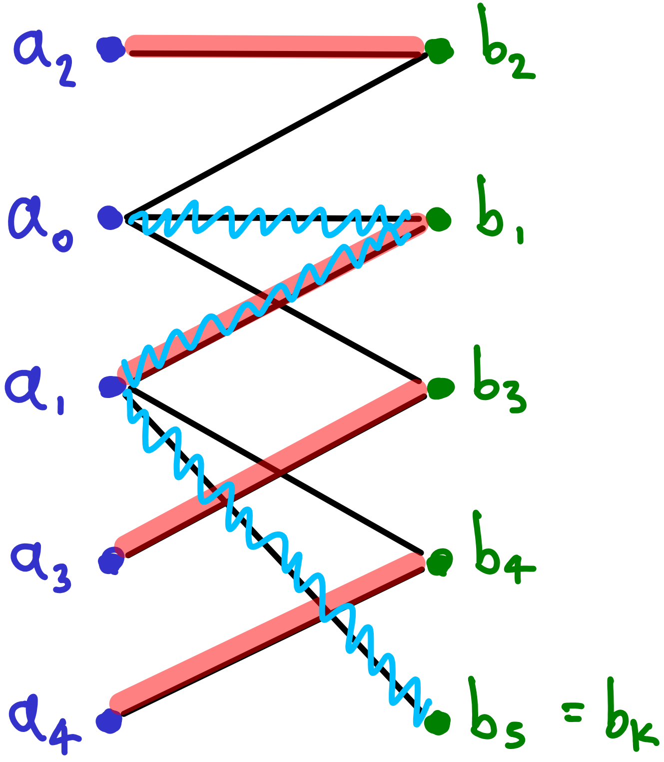
& we've only used $b_1 \ldots b_k$

$\exists$ some other $b \neq b_1 \ldots b_k$ in $N(a_0 \ldots a_k)$

If a_i goes to $b_{i \neq k}$ then $\exists$ match $b_i a_i$

Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



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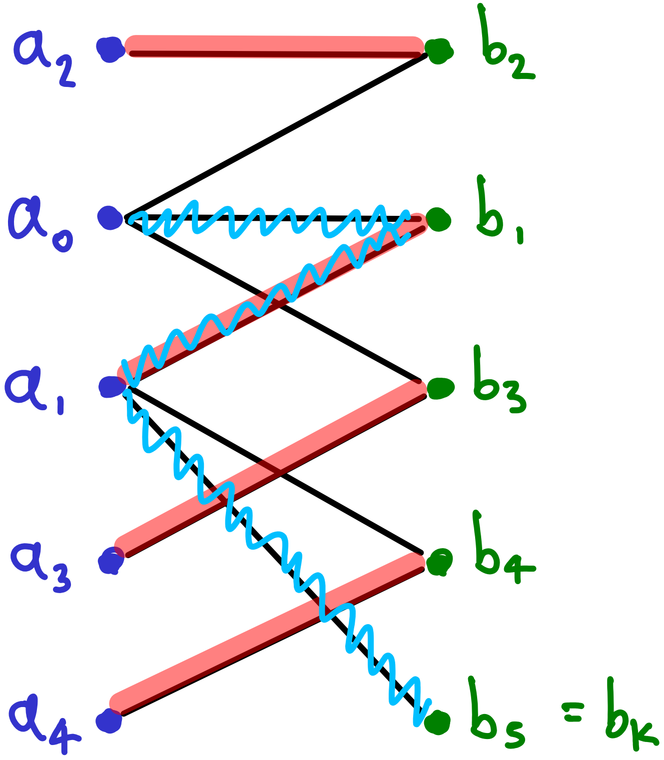
or

~~$a_0 b_1 a_1 b_2 a_2 \dots a_K$?~~

$a_0 \underbrace{b_1 a_1} \underbrace{b_2 a_2} \underbrace{b_3 a_3} \underbrace{b_4 a_4} \underbrace{b_5 a_5} \underbrace{b_6 a_6} \underbrace{b_7 a_7} \dots \underbrace{b_{K-1} a_{K-1}} b_K$

Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



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$a_0 b_1 a_1 b_2 a_2 \ldots b_k$

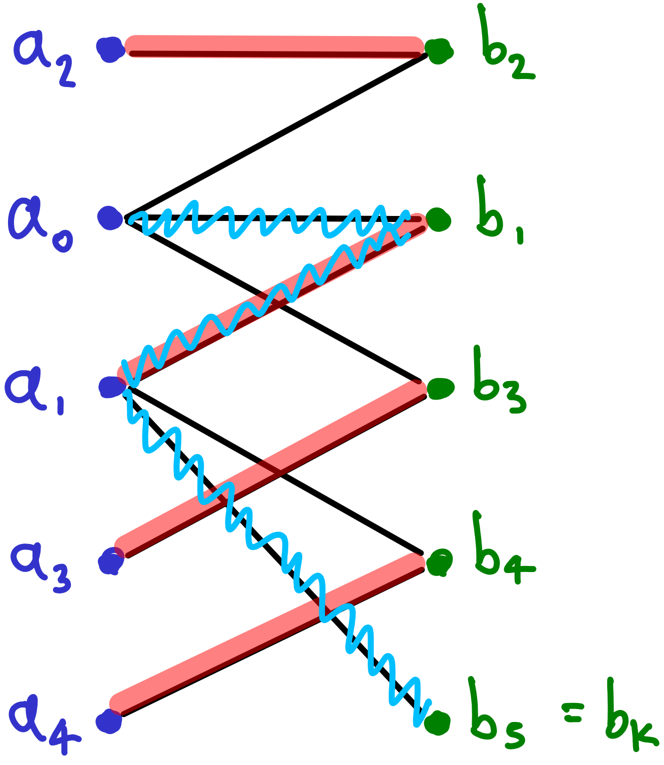
or

~~$a_0 b_1 a_1 b_2 a_2 \ldots a_k$?~~

$a_0 b_1 a_1 b_2 a_2 b_3 a_3 b_4 a_4 b_5 a_5 b_6 a_6 b_7 a_7 \ldots b_{k-1} a_{k-1} b_k$

Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



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& we've only used $b_1 \dots b_k$

$\exists$ some other $b \neq b_1 \dots b_k$ in $N(a_0 \dots a_k)$

$\boxed{\exists \text{ alternating path: } b_k \dots a_0}$

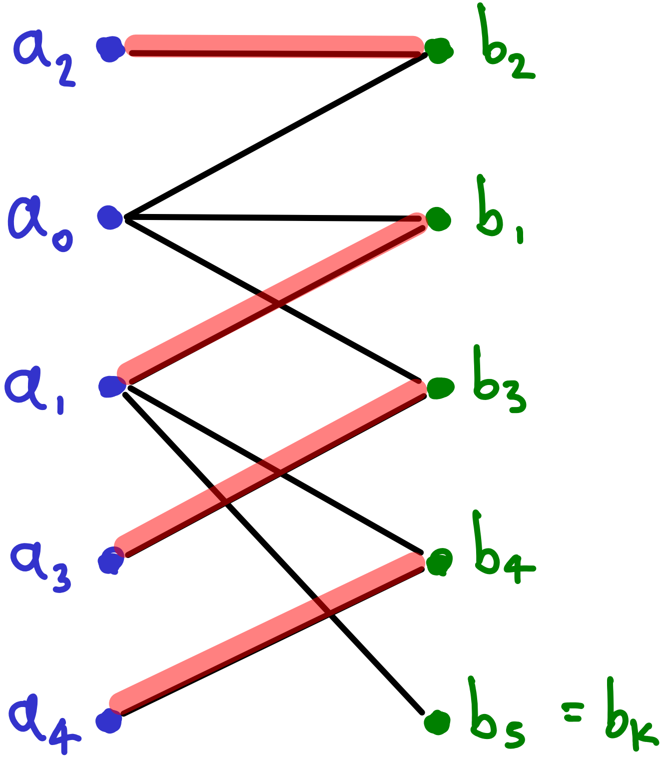
$a_0 b_1 a_1 b_2 a_2 \dots b_k$

or

$\cancel{a_0 b_1 a_1 b_2 a_2 \dots a_k\ ?}$

Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



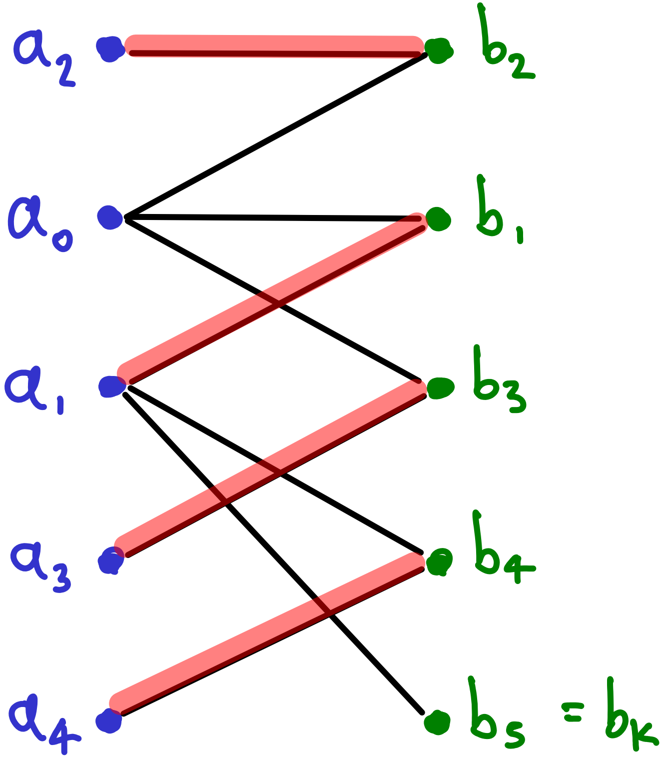
b_k doesn't match to any $a_0 \dots a_{k-1}$

by definition (each such a_j matched b_j)

(& a_0 trivially)

$$a_0 \underbrace{b_1 a_1} \underbrace{b_2 a_2} \dots b_k$$

Start w/ best matching. Suppose $|N(S)| \geq |S|$ but a_0 unmatched



b_K doesn't match to any $a_0 \dots a_{K-1}$

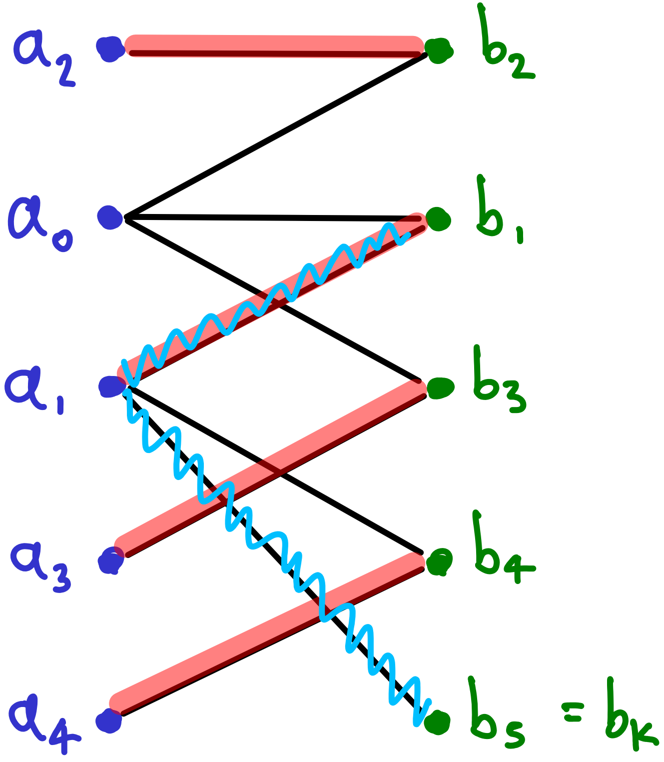
by definition (each such a_j matched b_j)

& doesn't match to any $a \neq a_0 \dots a_{K-1}$

because we would extend the sequence

$$a_0 \underbrace{b_1 a_1} \underbrace{b_2 a_2} \dots b_K$$

Start w/ best matching.



$a_0 b_1 a_1 b_2 a_2 \dots b_k$

Suppose $|N(S)| \geq |S|$ but a_0 unmatched

b_k doesn't match to any $a_0 \dots a_{k-1}$

by definition (each such a_j matched b_j)

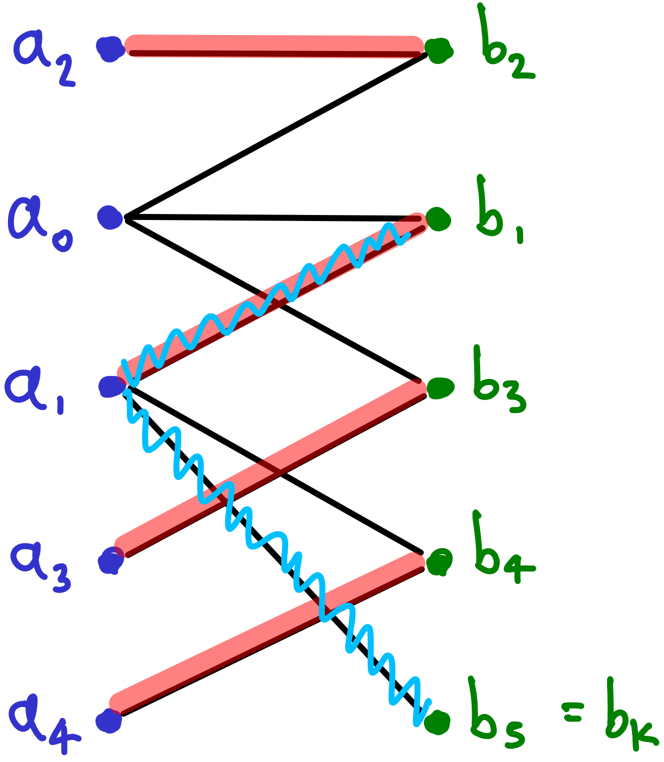
& doesn't match to any $a \neq a_0 \dots a_{k-1}$

because we would extend the sequence

$b_k \to$ some a_i $(i<k)$ $\to b_i$

Start w/ best matching.

Suppose $|N(S)| \geq |S|$ but a_0 unmatched



$a_0 b_1 a_1 b_2 a_2 \ldots b_K$

b_K doesn't match to any $a_0 \ldots a_{K-1}$

by definition (each such a_j matched b_j)

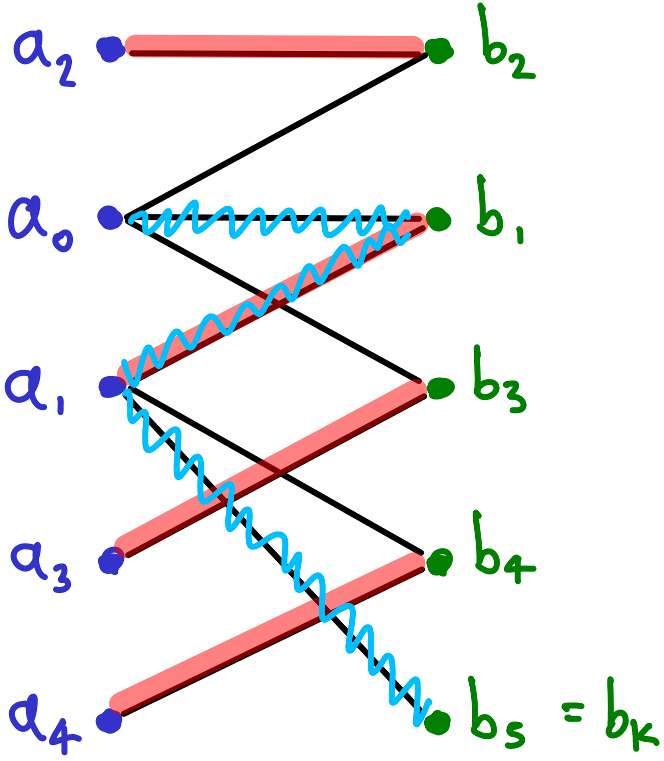
& doesn't match to any $a \neq a_0 \ldots a_{K-1}$

because we would extend the sequence

$b_K \to$ some a_i $(i<k) \to b_i \to$ some a_j $\underbrace{(j<i)}$

because b_i was discovered by some $a_{j<i}$

Start w/ best matching. *



$a_0 b_1 a_1 b_2 a_2 \ldots b_K$

Suppose $|N(S)| \geq |S|$ but a_0 unmatched

b_K doesn't match to any $a_0 \ldots a_{K-1}$

by definition (each such a_j matched b_j)

& doesn't match to any $a \neq a_0 \ldots a_{K-1}$

because we would extend the sequence

$b_K \to$ some a_i $(i<k) \to b_i \to$ some a_j $(j<i) \to b_j \to$ etc $\to a_0$

AUGMENTING PATH :

contradict *

□