ALL-PAIRS SHORTEST PATHS

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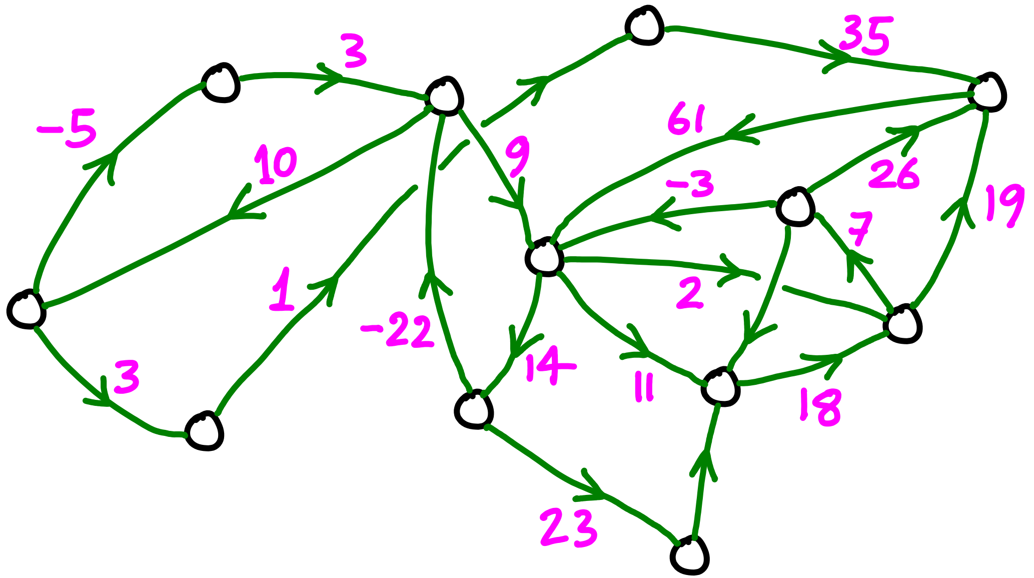
Input: weighted graph

• assume no negative cycles



Output: min-weight path for every pair of vertices

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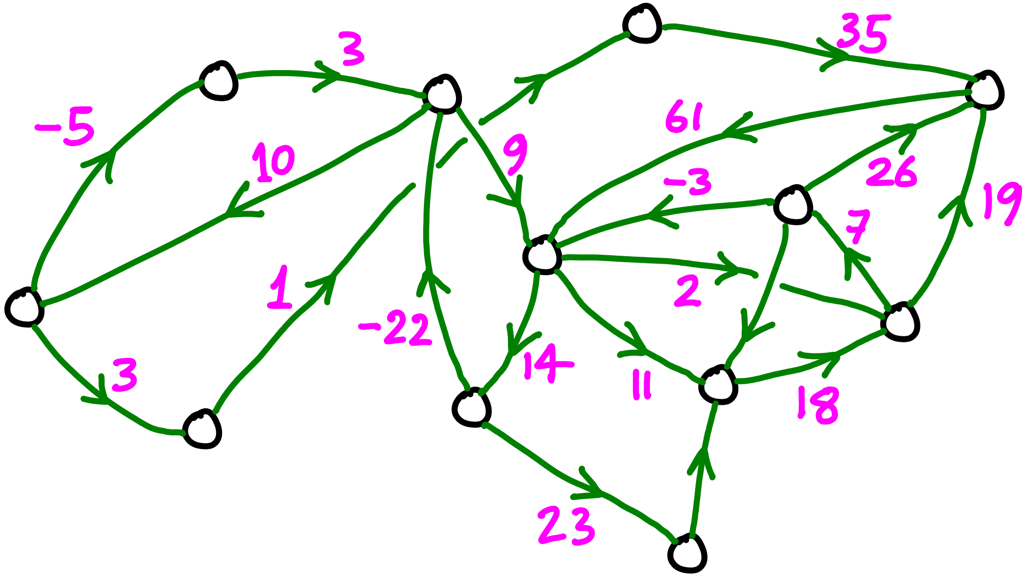

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Size of output?

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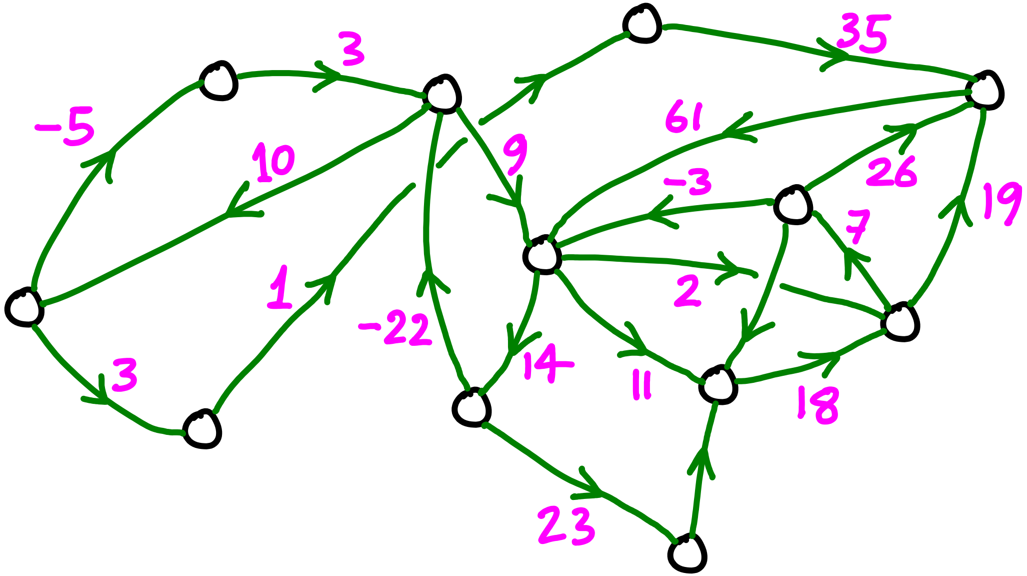

Output: min-weight path for $\underbrace{\text{every pair of vertices}}_{\Theta(V^2)}$

Size of output? → $\Theta(V^2) \cdot \underbrace{O(V)}_{\text{path length}}$

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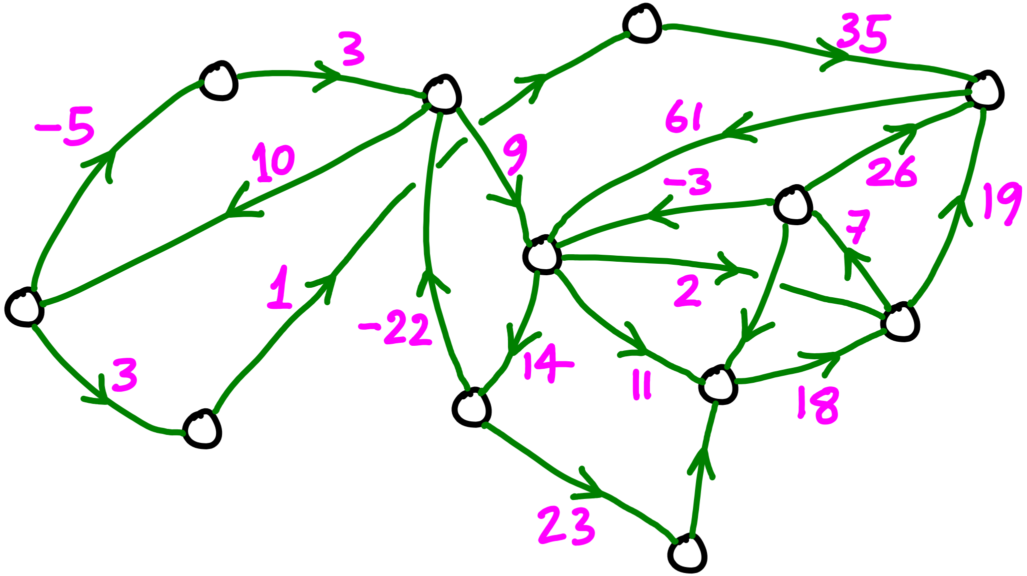

Output: min-weight path for $\underbrace{\text{every pair of vertices}}$

Size of output? $\longrightarrow$ $\Theta(V^2) \cdot \underbrace{O(V)}_{\text{path length}}$

↓

$\Theta(V^2)$: for every pair (x,y) store path weight & pred(y) (like tree for SSSP)

Intuitive solution ?

Intuitive solution: run SSSP, V times (once per vertex-source)

Cost ?

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Cost: $V \cdot O(V \cdot E)$ w/ B-F

...

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(adj. list)

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* $V \cdot O(E \log V)$ w/ bin. heap Dijkstra (adj. list)

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* $V \cdot O(E + V \log V)$ w/ Fibonacci heap Dijkstra (adj. list)

Intuitive solution: run SSSP, V times (once per vertex-source)

Cost: $V \cdot O(V \cdot E)$ w/ B-F

require weights ≥ 0

$V \cdot O(E \log V)$ w/ bin.heap Dijkstra (adj. list)

$V \cdot \Theta(V^2)$ w/ "array" Dijkstra (adj. matrix)

$V \cdot O(E + V \log V)$ w/ Fibonacci heap Dijkstra (adj. list)

To handle negative weights, all we get is $O(V^2 E) = O(V^4)$

Recall from SSSP: • if no negative cycles, shortest path length $\leq V-1$

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incremental: compute shortest path $i \rightsquigarrow j$ with length $\leq m$

$(m = 1 \ldots V-1)$

Recall from SSSP: • if no negative cycles, shortest path length $\le V-1$

• shortest paths contain shortest subpaths. $i \rightsquigarrow k \rightsquigarrow j$

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$$i \overset{\le m-1}{\rightsquigarrow} k \xrightarrow{w_{kj}} j$$

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trivially we know: shortest path $i \rightsquigarrow j$ with length ≤ 1 $\qquad \ell_{ij}^{1} = w_{ij}$

incremental: compute shortest path $i \rightsquigarrow j$ with length $\le m$ $\qquad \ell_{ij}^{m}$

$(m = 1 \ldots V-1)$

$$\ell_{ij}^{m} = \,?$$

$$i \xrightarrow{\le m-1} k \xrightarrow{w_{kj}} j$$

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$(m = 1 \ldots V-1)$

$$\ell_{ij}^{m} = \quad \cdots \quad \min_{1 \le k \le V} \left(\ell_{ik}^{m-1} + w_{kj} \right)$$

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Recall from SSSP: • if no negative cycles, shortest path length ≤ V-1

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trivially we know: shortest path $i \rightsquigarrow j$ with length ≤ 1 $\ell_{ij}^{1} = w_{ij}$

incremental: compute shortest path $i \rightsquigarrow j$ with length ≤ m ℓ_{ij}^{m}
$(m = 1 \ldots V-1)$

$$\ell_{ij}^{m} = \min\left\{ \ell_{ij}^{m-1}, \min_{1 \le k \le V} \left(\ell_{ik}^{m-1} + w_{kj} \right) \right\}$$

$i \overset{\le m-1}{\rightsquigarrow} k \xrightarrow{w_{kj}} j$

$$l_{ij}^{m} = \min\left\{ l_{ij}^{m-1} ,\ \min_{1 \le k \le V} \left(l_{ik}^{m-1} + w_{kj} \right) \right\}$$

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$$L^{m} \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix} \text{ from } L^{m-1} \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix} \ \& \ W \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$$

(input)

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Time:

?

Dynamic programming

(input)

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$$i \overset{\le m-1}{\rightsquigarrow} k \to j$$

Time:

V to compute $+$

$$\ell_{ij}^{m} = \min_{1\le k\le V} \left\{ \ell_{ik}^{m-1} + w_{kj} \right\}$$

Dynamic programming

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Time:

V^2 elements

↳ V to compute

Dynamic programming

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Time: $\Theta(V^4)$

~V matrices, $L^1 \dots L^{V-1}$

↳ V^2 elements each

↳ V to compute +

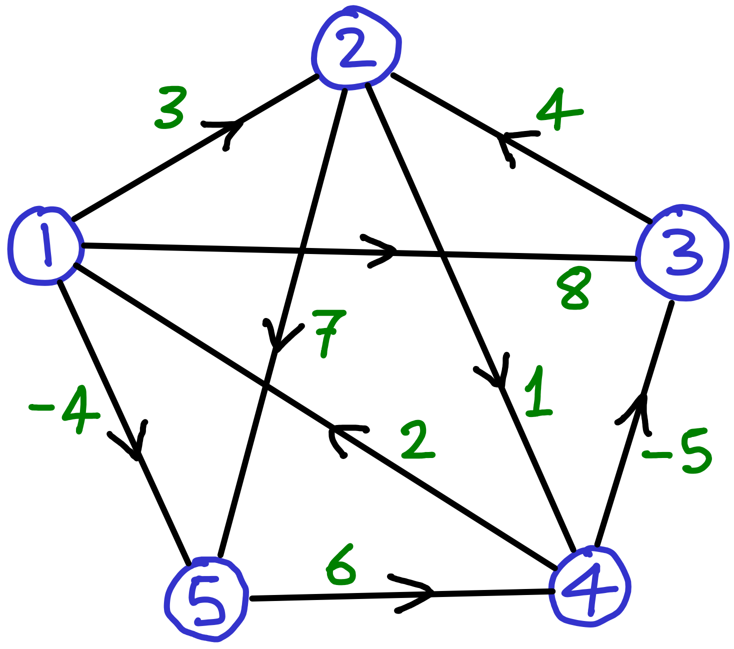
Dynamic programming

$$L^m \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix} \text{ from } L^{m-1} \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix} \ \& \ W \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$$

(input)

$$W = L^1 = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

Example



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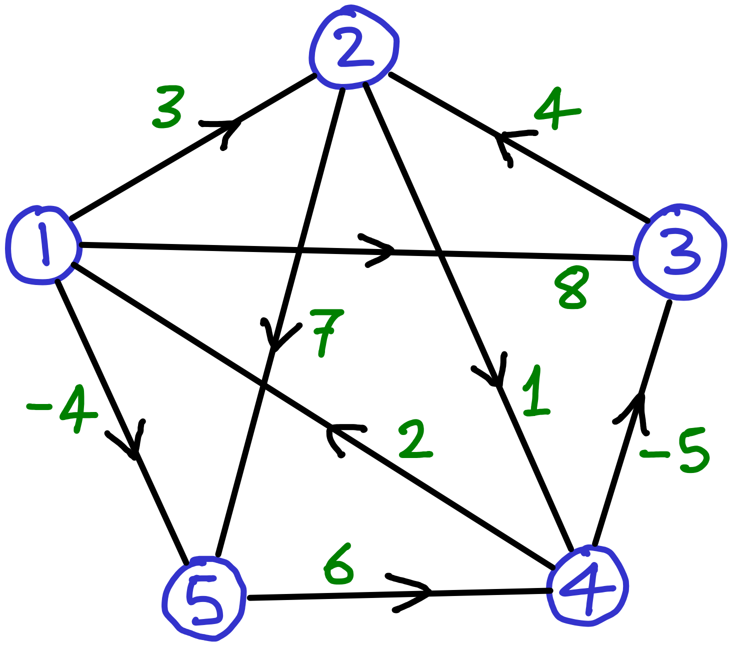
L^2

$i=4$?

$j=4$

$$\min\left\{ \ell^1_{4k} + w_{k4} \right\}$$



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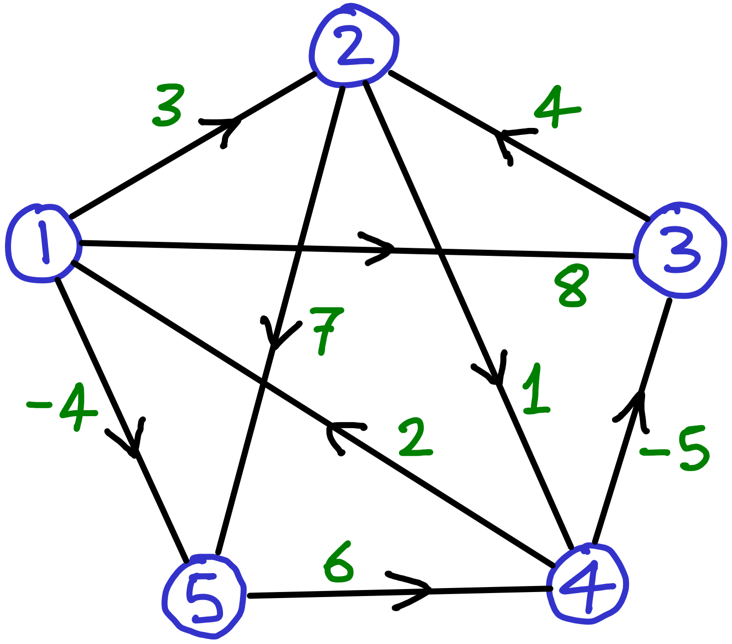
$$L^2 \qquad \begin{array}{c|c} & \\ i=4 & \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & 0 & \\ & & & & \end{bmatrix} \\ & j=4 \end{array}$$

$$\min\left\{ \ell^1_{4k} + w_{k4} \right\}$$

$$\min\left\{ 2+\infty,\ \infty+1,\ -5+\infty,\ 0+0,\ \infty+6 \right\}$$

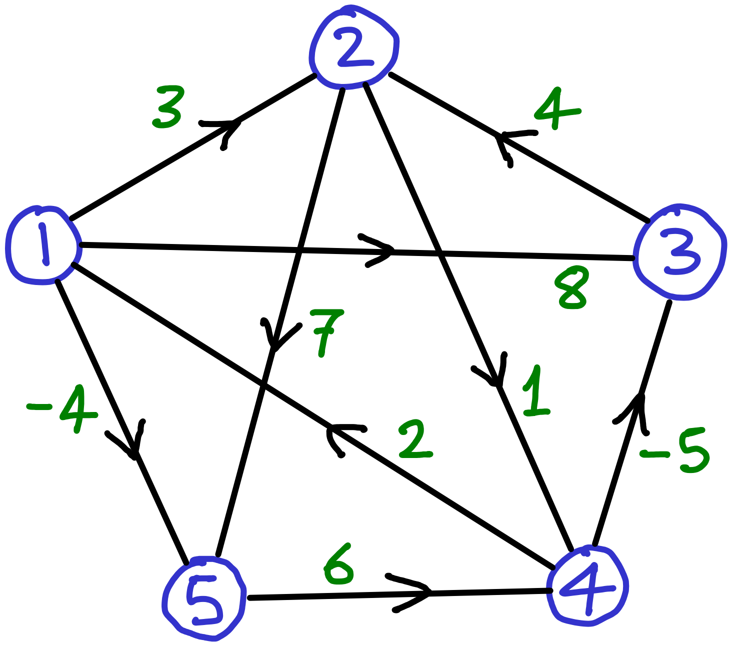
$$\ell^m_{ij} = \min_{1 \le k \le v}\left\{ \ell^{m-1}_{ik} + w_{kj} \right\}$$



$$\begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} W = L^1$$

$$L^2 \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}$$

Why?



$$l_{ij}^{m} = \min_{1 \le k \le v} \left\{ l_{ik}^{m-1} + w_{kj} \right\}$$

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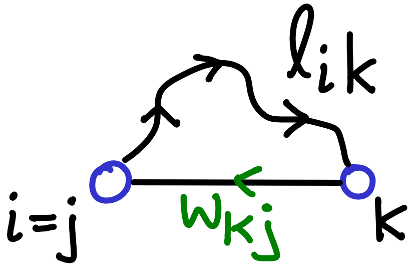
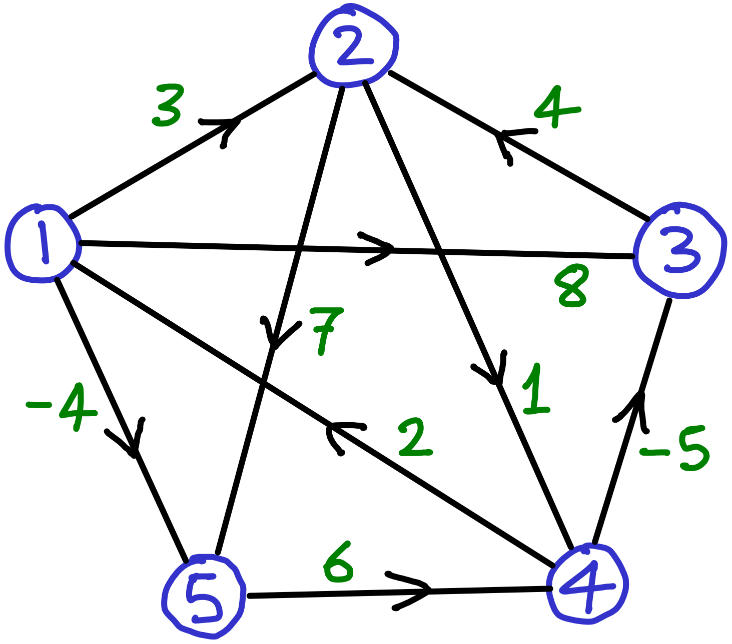
$$L^2 = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}$$

Why?

No negative cycles





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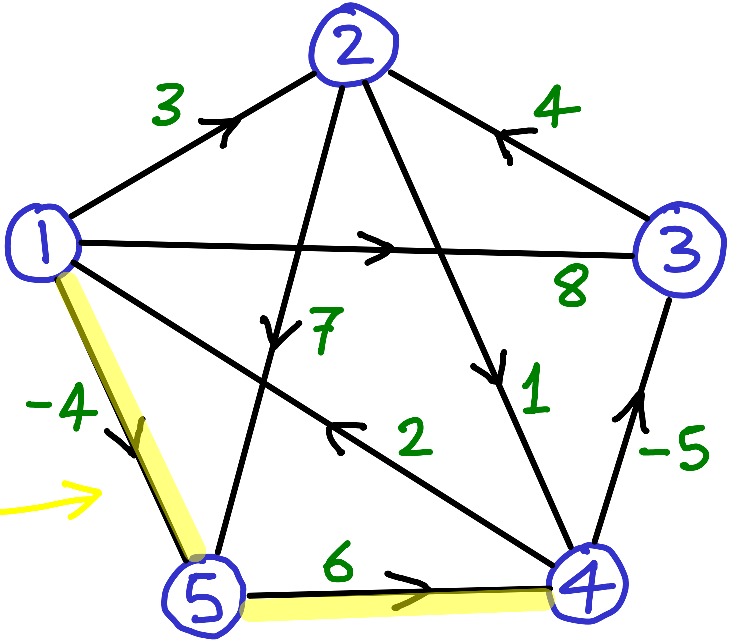
\begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} W = L^1
\qquad
L^2 \begin{bmatrix} 0 & & & 2 & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}
$$

$$\min \left\{ l^1_{1k} + w_{k4} \right\}$$

$$\min \left\{ 0+\infty,\ 3+1,\ 8+\infty,\ \infty+0,\ -4+6 \right\}$$

$$l^m_{ij} = \min_{1 \le k \le V} \left\{ l^{m-1}_{ik} + w_{kj} \right\}$$



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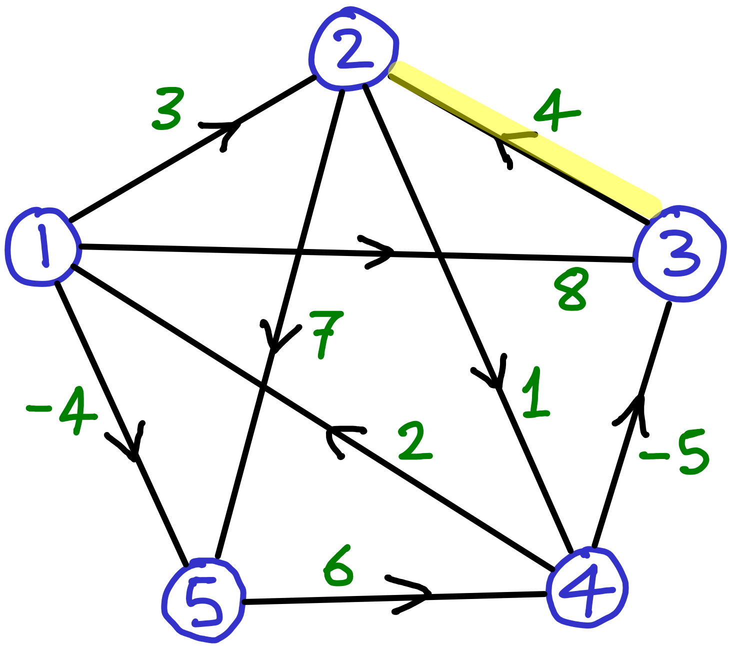
$$L^2 = \begin{bmatrix} 0 & & & 2 & \\ & 0 & & & \\ & 4 & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}$$

$$\min \left\{ l^1_{3k} + w_{k2} \right\}$$

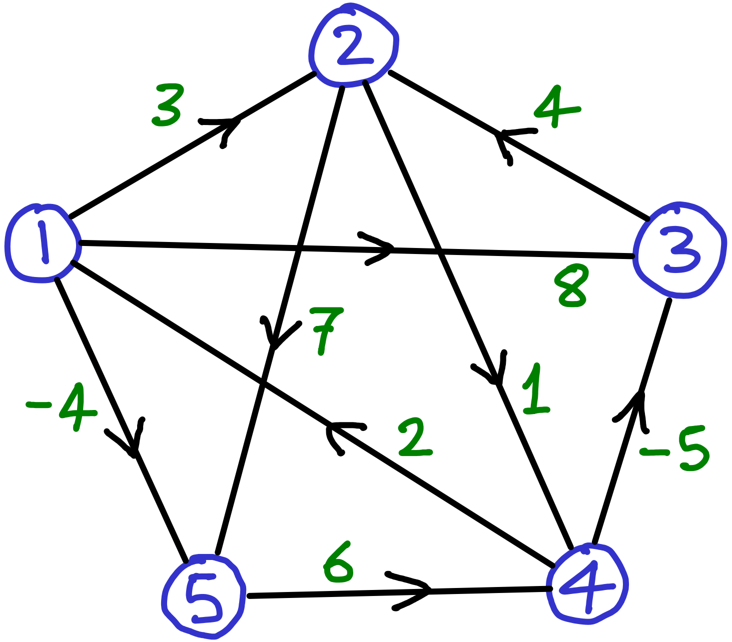
$$\min \left\{ \infty + 3,\ 4 + 0,\ 0 + 4,\ \infty + \infty,\ \infty + \infty \right\}$$

$$l^m_{ij} = \min_{1 \le k \le v} \left\{ l^{m-1}_{ik} + w_{kj} \right\}$$



$$\begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} W = L^1$$

$$L^2 \begin{bmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{bmatrix}$$



$$l_{ij}^{m} = \min_{1 \le k \le v} \left\{ l_{ik}^{m-1} + w_{kj} \right\}$$

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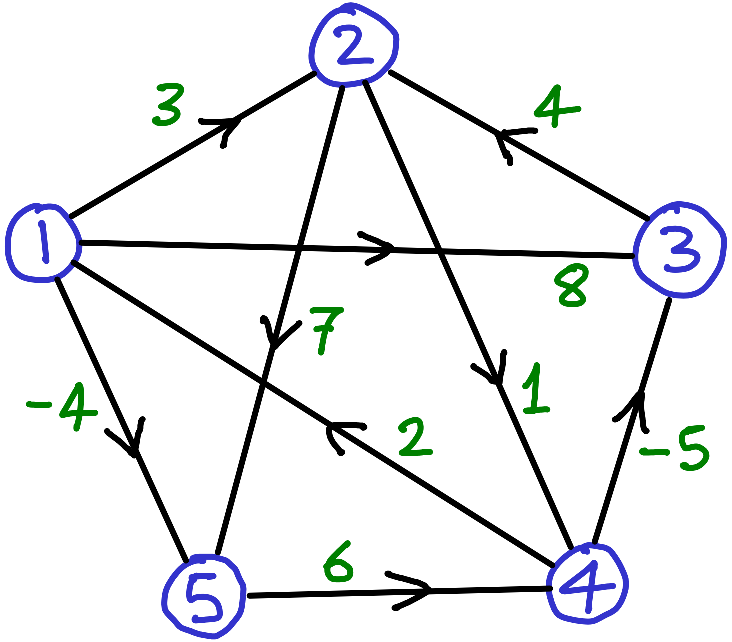
$$L^3 = \begin{array}{c} \\ \\ \\ \\ i=5 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & ? & & & 0 \end{bmatrix} \\ \begin{array}{ccccc} & j=2 & & & \end{array} \end{array}$$



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$$
\begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} W = L^1
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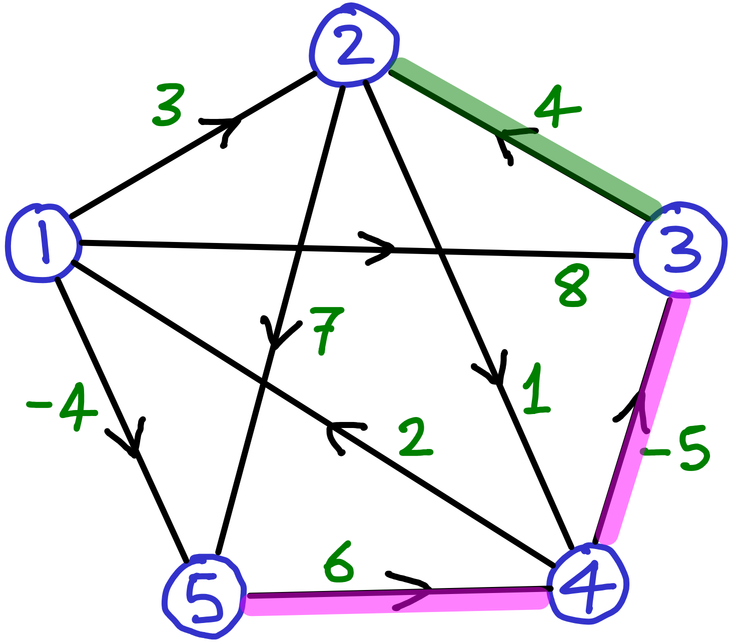
$$

$$
L^3 \quad
\begin{array}{c c}
 & \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & 5 & & & 0 \end{bmatrix} \\
i=5 & \\
 & j=2
\end{array}
$$

$$\min \left\{ l^2_{5K} + w_{K2} \right\}$$

$$\min \left\{ 8+3,\ \infty+0,\ 1+4,\ 6+\infty,\ 0+\infty \right\}$$

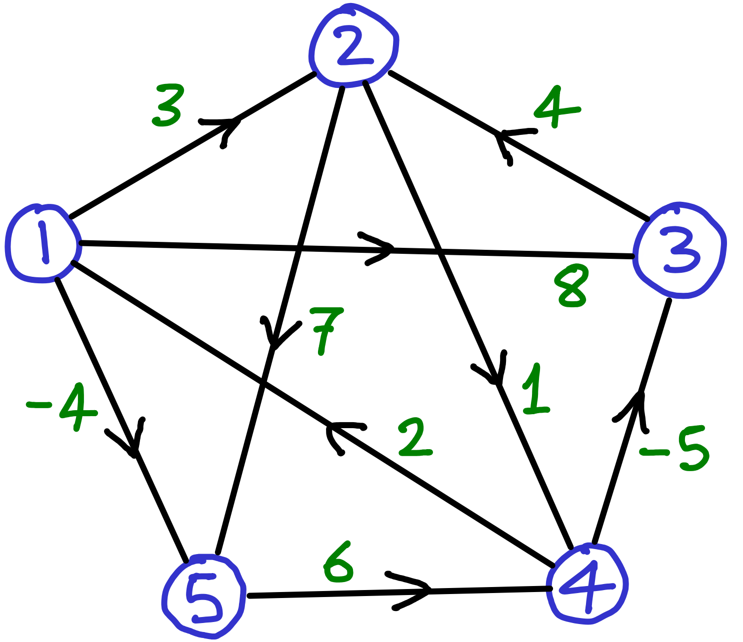
$$l^m_{ij} = \min_{1 \le k \le v} \left\{ l^{m-1}_{ik} + w_{kj} \right\}$$



$$\begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} W = L^1$$

$$L^2 \begin{bmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{bmatrix}$$

$$L^3 \begin{bmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$



$$l_{ij}^{m} = \min_{1 \le k \le v} \left\{ l_{ik}^{m-1} + w_{kj} \right\}$$

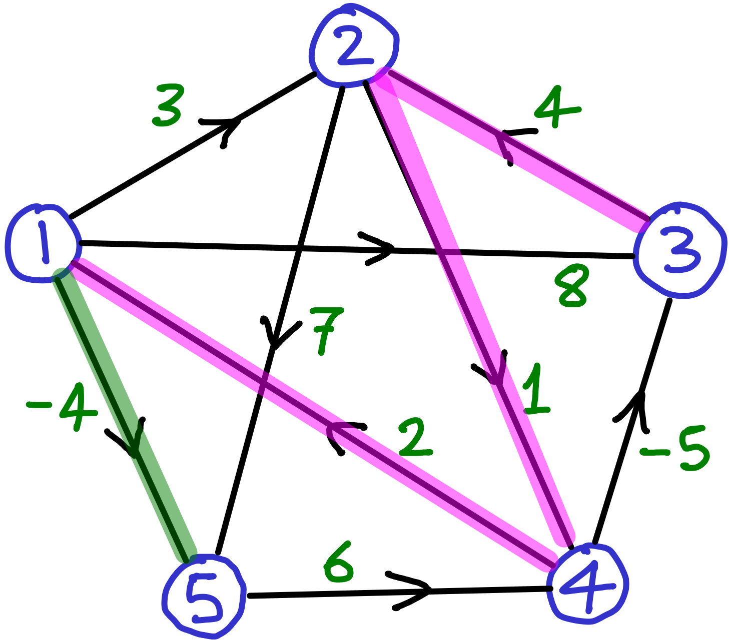
$$\begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \quad W = L^1$$

$$L^2 \begin{bmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{bmatrix}$$

$$L^3 \begin{bmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$

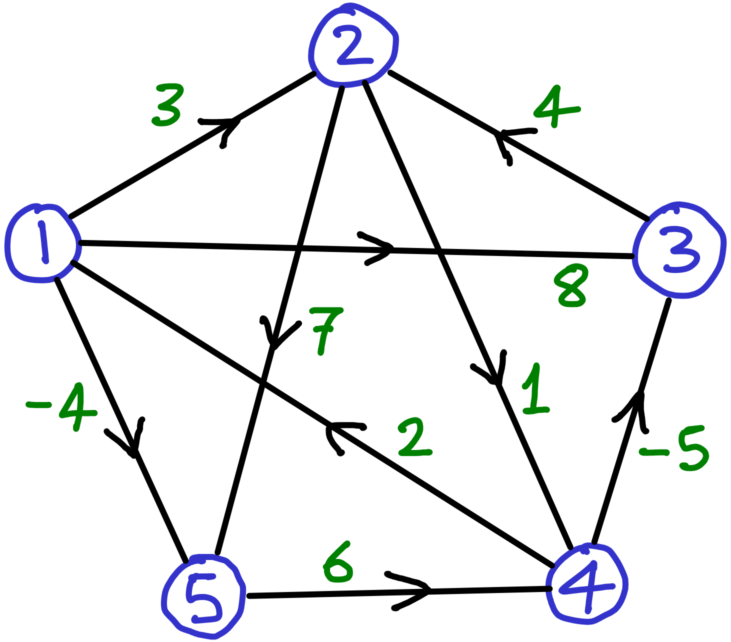
$$L^4 \begin{bmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$

$$\ell_{ij}^{m} = \min_{1 \le k \le v} \Big\{ \overbrace{\ell_{ik}^{m-1}}^{3 \to 1} + \overbrace{w_{kj}}^{1 \to 5} \Big\}$$



$$
W = L^1 = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}
\qquad
L^2 = \begin{bmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{bmatrix}
\qquad
L^3 = \begin{bmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}
$$

$$
L^4 = \begin{bmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}
\qquad
L^5 = \begin{bmatrix} 0 & & & ? & \\ & 0 & & & \\ & & 0 & & \\ & ? & & 0 & \\ & & & & 0 \end{bmatrix}
$$



$$
l_{ij}^{m} = \min_{1 \le k \le V} \left\{ l_{ik}^{m-1} + w_{kj} \right\}
$$

$$\begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} W = L^1$$

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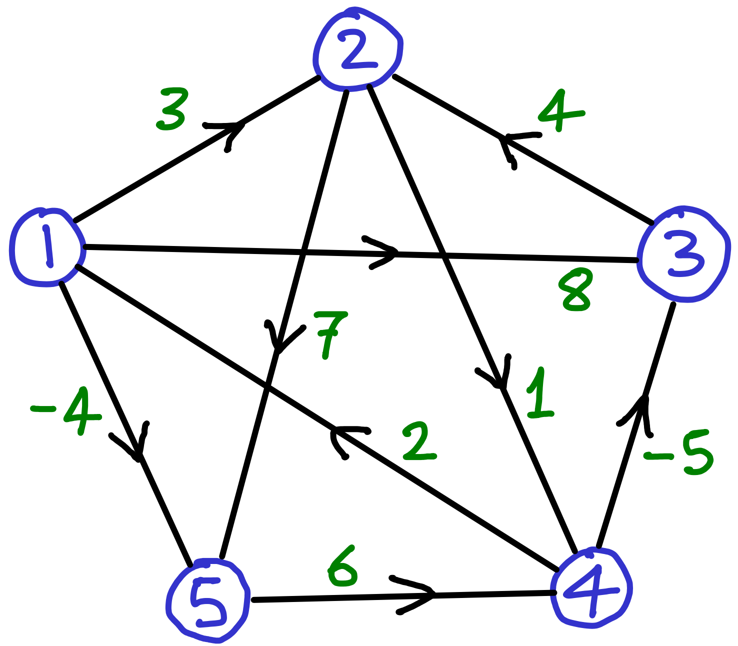
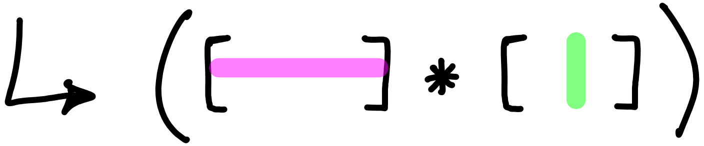
$$L^3 \begin{bmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$

$$L^4 = L^{V-1} \begin{bmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix} = L^V \begin{bmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$

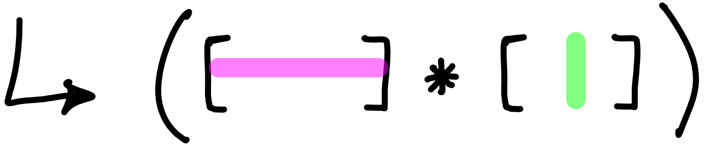


$$l_{ij}^{m} = \min_{1 \le k \le V} \left\{ l_{ik}^{m-1} + w_{kj} \right\}$$

$$l_{ij}^{m} = \min_{1 \le k \le v} \left\{ \underline{l_{ik}^{m-1}} + \underline{w_{kj}} \right\}$$



$$\ell_{ij}^{m} = \min_{1 \le k \le v} \left\{ \ell_{ik}^{m-1} + w_{kj} \right\}$$



matrix "product"

$$[L^{m}] = [L^{m-1}] * [W]$$

$$\ell_{ij}^{m} = \min_{1 \le k \le v} \left\{ \ell_{ik}^{m-1} + w_{kj} \right\}$$

matrix "product"

$$\hookrightarrow \left(\left[\quad \right] * \left[\quad \right] \right)$$

$$\left[L^{m} \right] = \left[L^{m-1} \right] * \left[W \right]$$

$$\ell_{ij}^{m} = \sum_{1 \le k \le v} \left\{ \ell_{ik}^{m-1} \cdot w_{kj} \right\}$$

$\leftarrow$ actual matrix multiplication

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← actual matrix multiplication

Not all matrix product results carry over (e.g. can't simulate Strassen's result)

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but this does: $[L^{m}] = [L^{m/2}] * [L^{m/2}]$

$$l_{ij}^{m} = \min_{1 \le k \le v} \{ l_{ik}^{m-1} + w_{kj} \}$$

matrix "product"

$$\hookrightarrow \left([\quad] * [\quad] \right) \qquad [L^m] = [L^{m-1}] * [W]$$

$$l_{ij}^{m} = \sum_{1 \le k \le v} \{ l_{ik}^{m-1} \cdot w_{kj} \}$$

$\leftarrow$ actual matrix multiplication

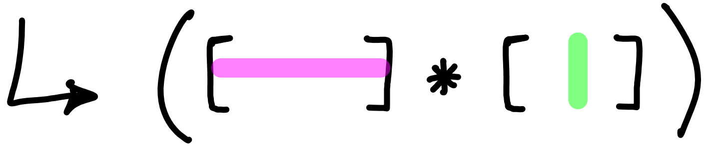
Not all matrix product results carry over (e.g. can't simulate Strassen's result)

but this does: $[L^m] = [L^{m/2}] * [L^{m/2}]$

$L^2 = W \cdot W = W^2$

$$\ell_{ij}^{m} = \min_{1 \le k \le v} \left\{ \ell_{ik}^{m-1} + w_{kj} \right\}$$

matrix "product"



$$[L^{m}] = [L^{m-1}] * [W]$$

$$\ell_{ij}^{m} = \sum_{1 \le k \le v} \left\{ \ell_{ik}^{m-1} \cdot w_{kj} \right\}$$

$\leftarrow$ actual matrix multiplication

Not all matrix product results carry over (e.g. can't simulate Strassen's result)

but this does: $[L^{m}] = [L^{m/2}] * [L^{m/2}]$

$L^2 = W \cdot W = W^2$ $\qquad$ $L^4 = L^2 \cdot L^2 = W^4$

$$\ell_{ij}^{m} = \min_{1 \le k \le v} \{ \ell_{ik}^{m-1} + w_{kj} \}$$

matrix "product"

$$[L^m] = [L^{m-1}] * [W]$$

$$\ell_{ij}^{m} = \sum_{1 \le k \le v} \{ \ell_{ik}^{m-1} \cdot w_{kj} \}$$

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Not all matrix product results carry over (e.g. can't simulate Strassen's result)

but this does: $[L^m] = [L^{m/2}] * [L^{m/2}]$

$L^2 = W \cdot W = W^2$ // $L^4 = L^2 \cdot L^2 = W^4$ // $L^8 = L^4 \cdot L^4 = W^8$

"repeated squaring"

$$l_{ij}^{m} = \min_{1 \le k \le V} \{ l_{ik}^{m-1} + w_{kj} \}$$

matrix "product"

$$\hookrightarrow \left([\;\text{———}\;] * [\;|\;] \right)$$

$$[L^{m}] = [L^{m-1}] * [W]$$

$$l_{ij}^{m} = \sum_{1 \le k \le V} \{ l_{ik}^{m-1} \cdot w_{kj} \}$$

← actual matrix multiplication

Not all matrix product results carry over (e.g. can't simulate Strassen's result)

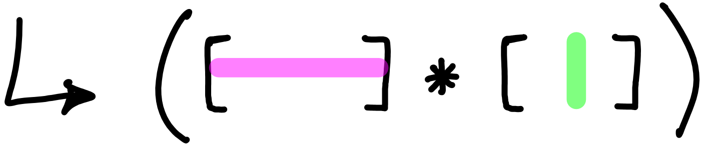
but this does: $[L^{m}] = [L^{m/2}] * [L^{m/2}]$

$L^2 = W \cdot W = W^2$ // $L^4 = L^2 \cdot L^2 = W^4$ // $L^8 = L^4 \cdot L^4 = W^8$

"repeated squaring"

After $\log V$ steps we get L^{k}, $k \ge V-1$.

$$\ell_{ij}^{m} = \min_{1 \le k \le V} \{ \ell_{ik}^{m-1} + w_{kj} \}$$ matrix "product"



$$[L^m] = [L^{m-1}] * [W]$$

$$\ell_{ij}^{m} = \sum_{1 \le k \le V} \{ \ell_{ik}^{m-1} \cdot w_{kj} \}$$ ← actual matrix multiplication

---

Not all matrix product results carry over (e.g. can't simulate Strassen's result)

but this does: $[L^m] = [L^{m/2}] * [L^{m/2}]$

$L^2 = W \cdot W = W^2$ // $L^4 = L^2 \cdot L^2 = W^4$ // $L^8 = L^4 \cdot L^4 = W^8$

"repeated squaring"

After $\log V$ steps we get L^K, $K \geq V-1$. Recall $L^K = L^{V-1}$ ⇒ DONE!

$$\ell_{ij}^{m} = \min_{1 \le k \le V} \left\{ \ell_{ik}^{m-1} + w_{kj} \right\}$$

matrix "product"

$$\hookrightarrow \left([\;\text{—}\;] * [\;|\;] \right) \qquad [L^{m}] = [L^{m-1}] * [W]$$

$$\ell_{ij}^{m} = \sum_{1 \le k \le V} \left\{ \ell_{ik}^{m-1} \cdot w_{kj} \right\}$$

$\leftarrow$ actual matrix multiplication

Not all matrix product results carry over (e.g. can't simulate Strassen's result)

but this does: $[L^{m}] = [L^{m/2}] * [L^{m/2}]$

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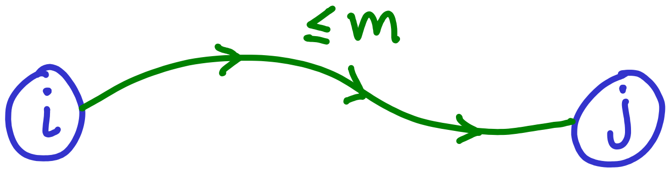
"repeated squaring"

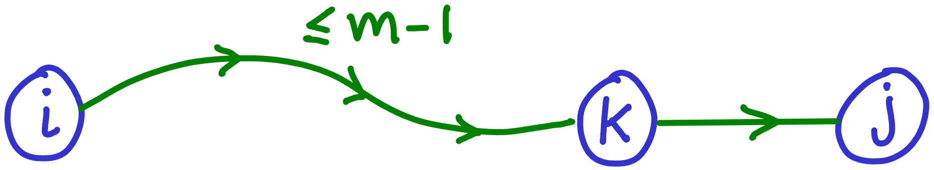
After $\log V$ steps we get L^{k}, $k \geq V-1$. Recall $L^{k} = L^{V-1}$ $\Rightarrow$ DONE!

Time: "product" $= \Theta(V^3)$ // # "products" $= O(\log V)$ // TOTAL $= \Theta(V^3 \log V)$

What we want for all i,j

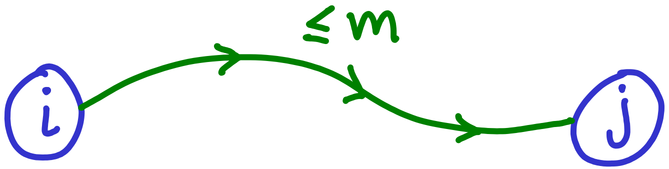


First dynamic programming approach

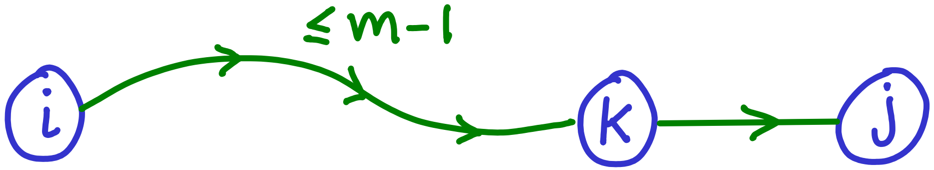


$\Theta(V^4)$

What we want for all i,j

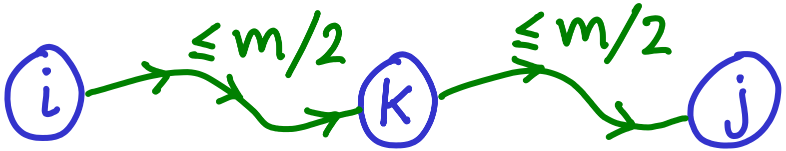


First dynamic programming approach

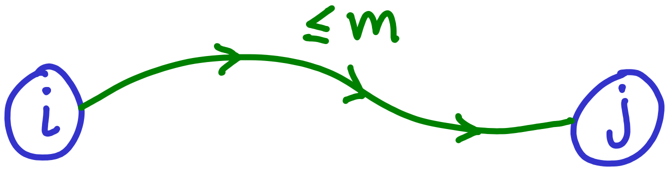


$\Theta(V^4)$

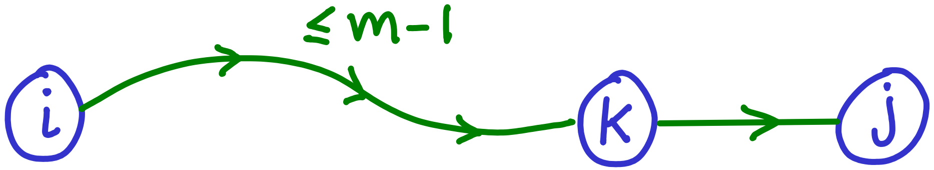
"Repeated squaring" dynamic programming



What we want for all i,j

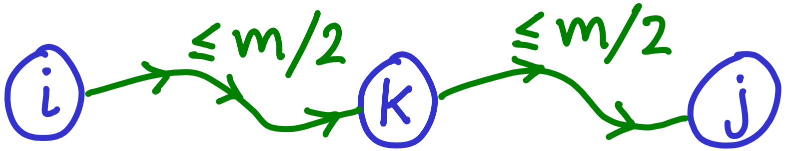


First dynamic programming approach



$\Theta(V^4)$

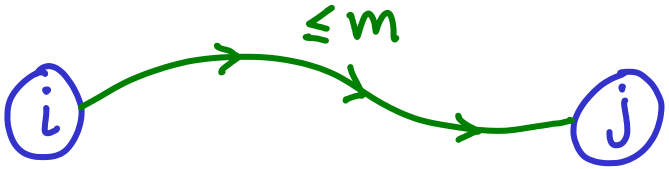
"Repeated squaring" dynamic programming



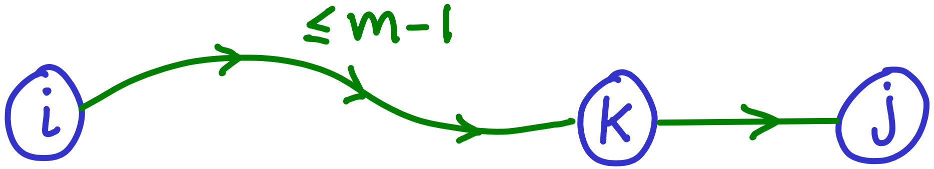
$$l_{ij}^{2m} = \min_{1 \le k \le V} \{ l_{ik}^{m} + l_{kj}^{m} \}$$

$\Theta(V^3 \log V)$

What we want for all i,j

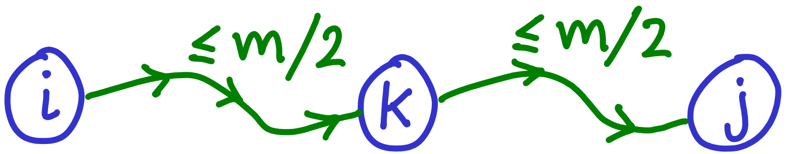


First dynamic programming approach



$\Theta(V^4)$

"Repeated squaring" dynamic programming



$$\ell_{ij}^{2m} = \min_{1 \le k \le V} \{ \ell_{ik}^{m} + \ell_{kj}^{m} \}$$

$\Theta(V^3 \log V)$

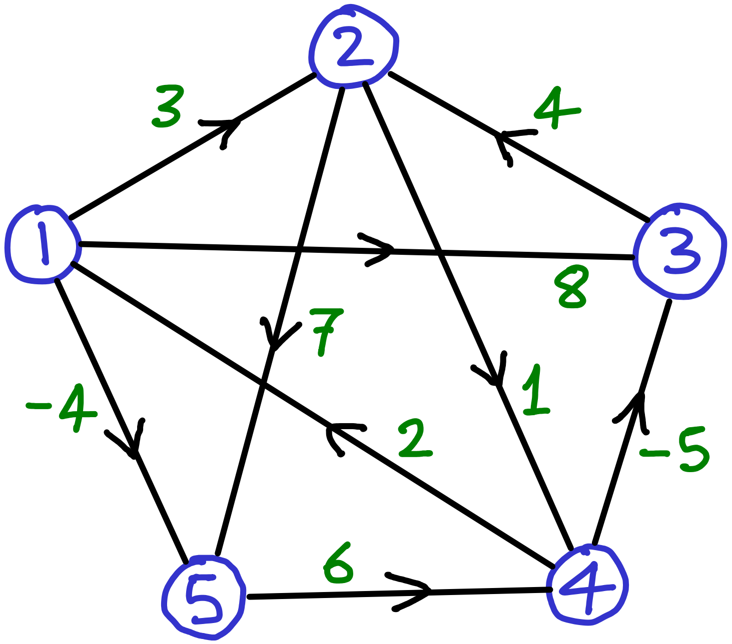
The matrix multiplication analogy isn't perfect but has been used to get several improvements. See CLRS chapter 25 notes (p.706)

FLOYD-WARSHALL ALGORITHM (Floyd-Roy-Kleene-Warshall)

FLOYD-WARSHALL ALGORITHM (Floyd-Roy-Kleene-Warshall)

$D^0 = W =$ all shortest paths $i \rightsquigarrow j$ without any intermediate stops

$$D^0 = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$



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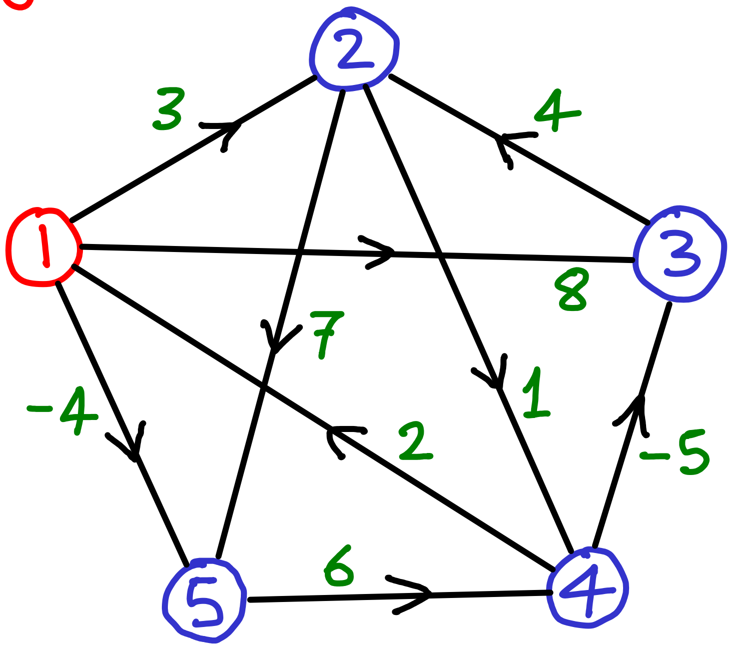
$D^0 = W =$ all shortest paths $i \rightsquigarrow j$ without any intermediate stops

$D^1 =$ all shortest paths $i \rightsquigarrow j$ with <u>at most 1</u> intermediate stop, specifically through v_1

$$D^0 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad D^1 \begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$$



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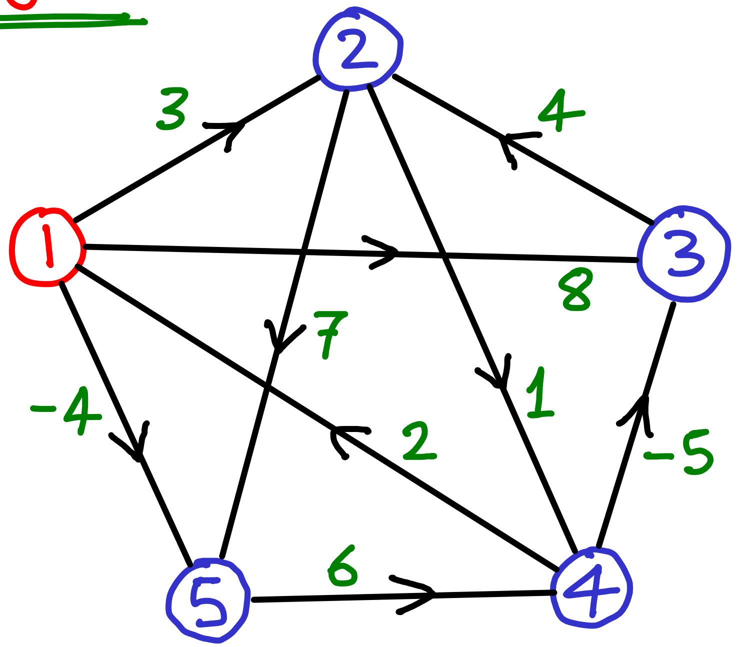

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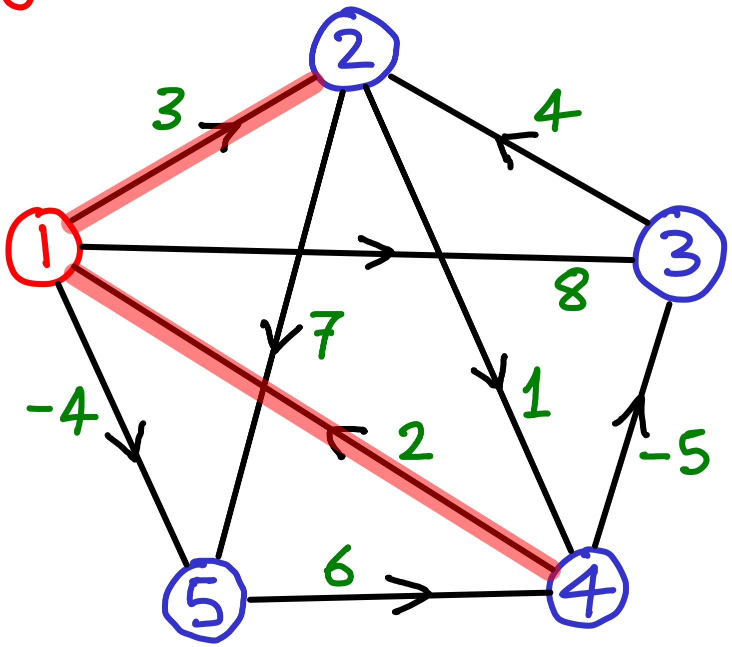
$$D^0 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^1 \quad \begin{array}{cc} & \begin{array}{ccccc} & 2 & & & \end{array} \\ \begin{array}{c} \\ \\ \\ 4 \\ \\ \end{array} & \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & & & & \\ \infty & & & & \\ 2 & 5 & & & \\ \infty & & & & \end{bmatrix} \end{array}$$



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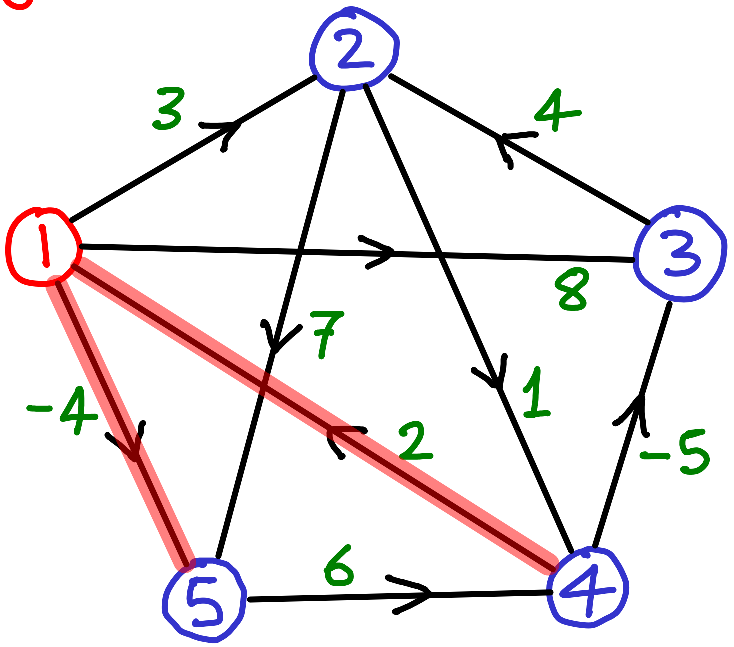
specifically through v_1

$$D^0 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^1 \quad \begin{array}{c} \\ \\ \\ \\ 4 \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} & & & & \quad\;\; 5 \end{array} \\ \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & & & & \\ \infty & & & & \\ 2 & 5 & & & -2 \\ \infty & & & & \end{bmatrix} \end{array}$$



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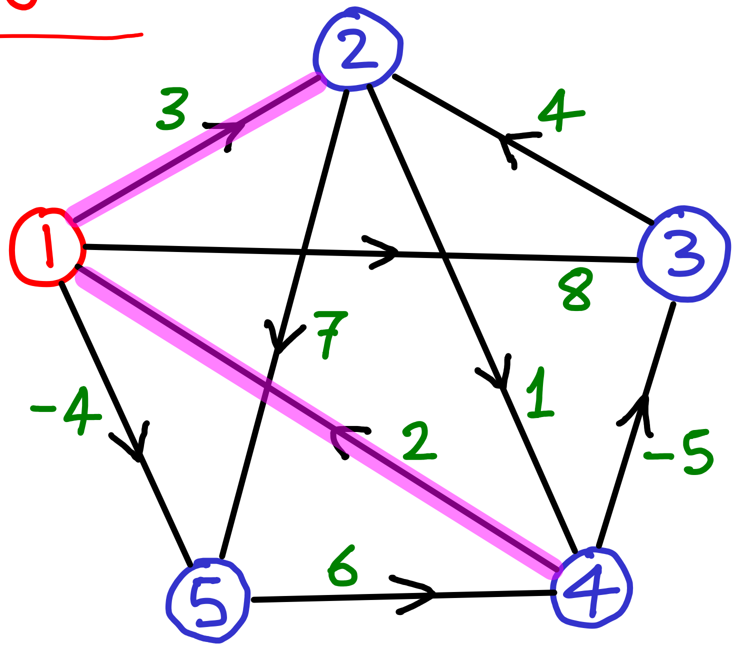
$D^0 = W =$ all shortest paths $i \rightsquigarrow j$ without any intermediate stops

$D^1 =$ all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, specifically through v_1

$$d^1_{4,2} = \min\{d^0_{4,2},\ d^0_{4,1} + d^0_{1,2}\}$$

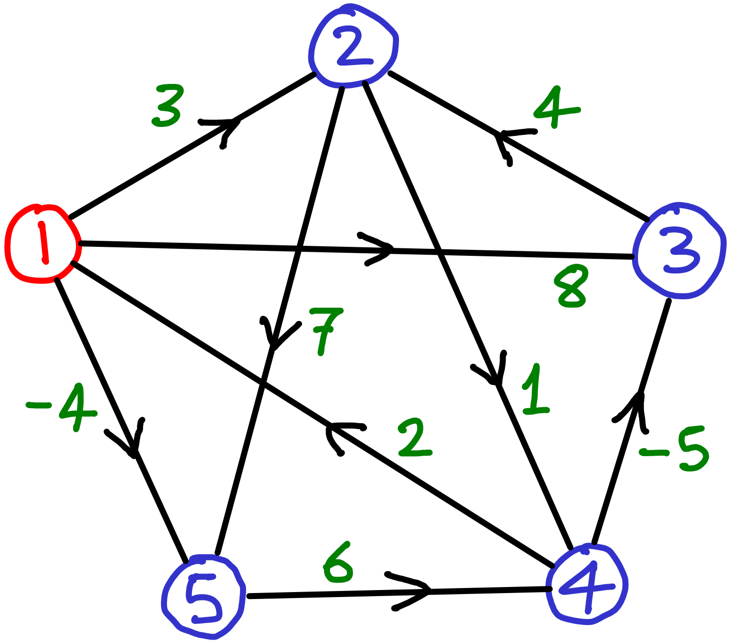
$$D^0 = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad D^1 = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$



D^1 = all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, via v_1

$$D^0 = W = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^1 = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

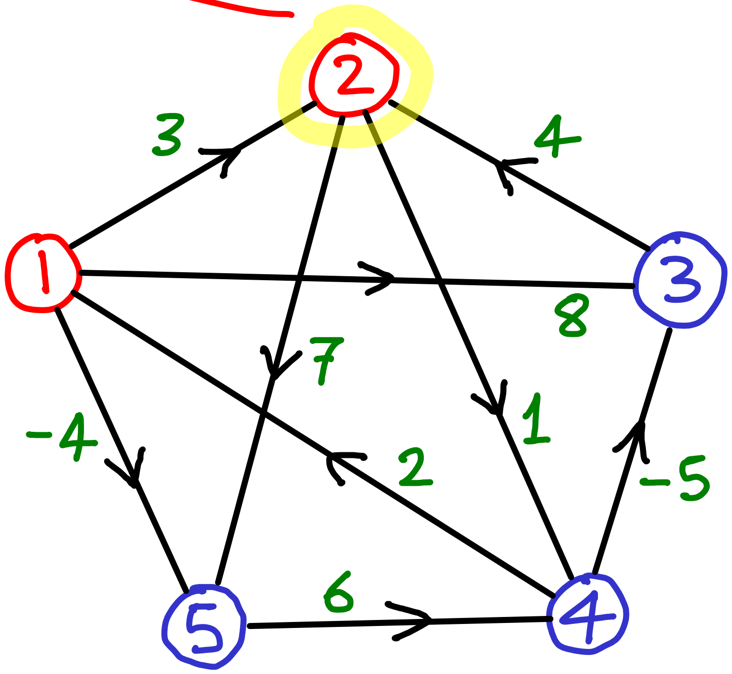


D^1 = all shortest paths $i \leadsto j$ with at most 1 intermediate stop, via V_1

Now let's allow V_2 to help

$$D^0 = W = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^1 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$



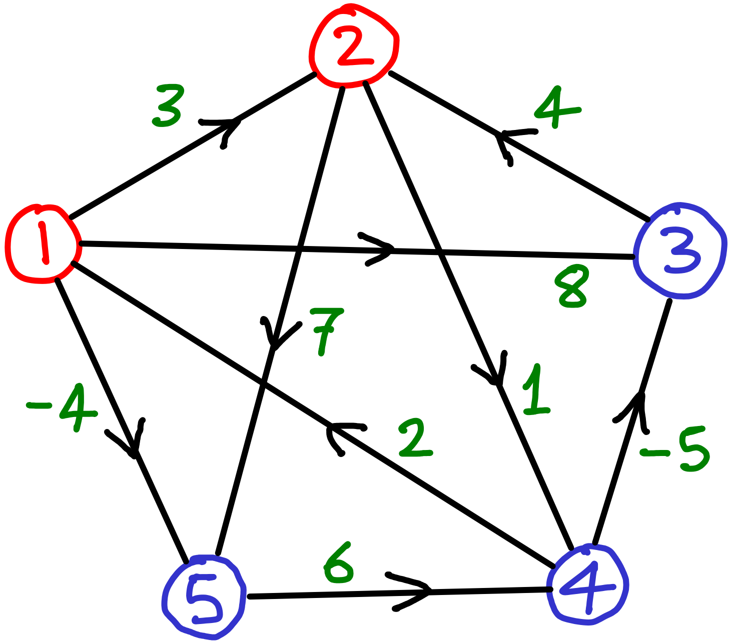
D^1 = all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, via V_1

Now let's allow V_2 to help

D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$

$$D^0 = W = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

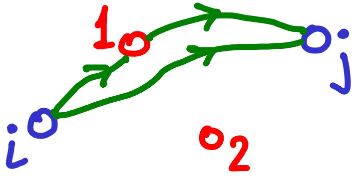
$$D^1 = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$



D^1 = all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, via V_1

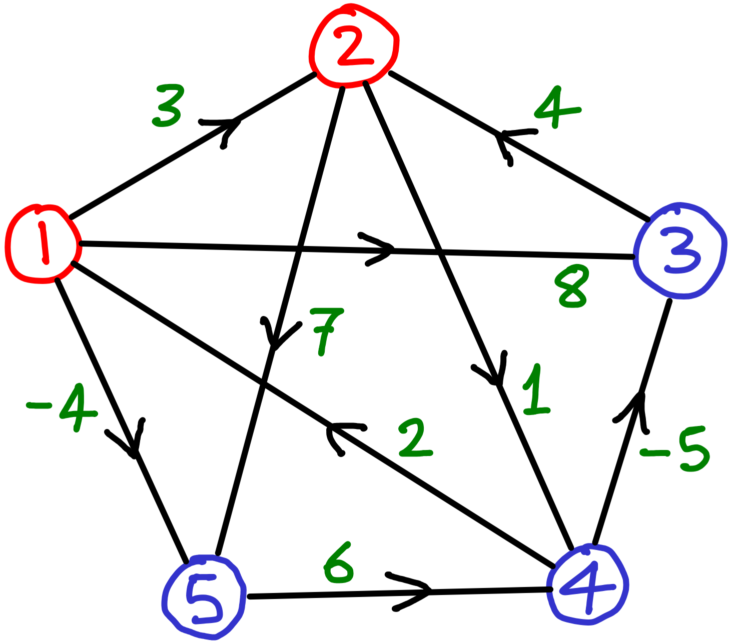
Now let's allow V_2 to help

D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$



$d^2_{i,j}$: if we don't use V_2 then solution is in $D^1 \Rightarrow d^1_{i,j}$

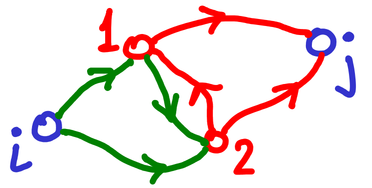
$$D^0 = W = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad D^1 = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$



D^1 = all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, via V_1

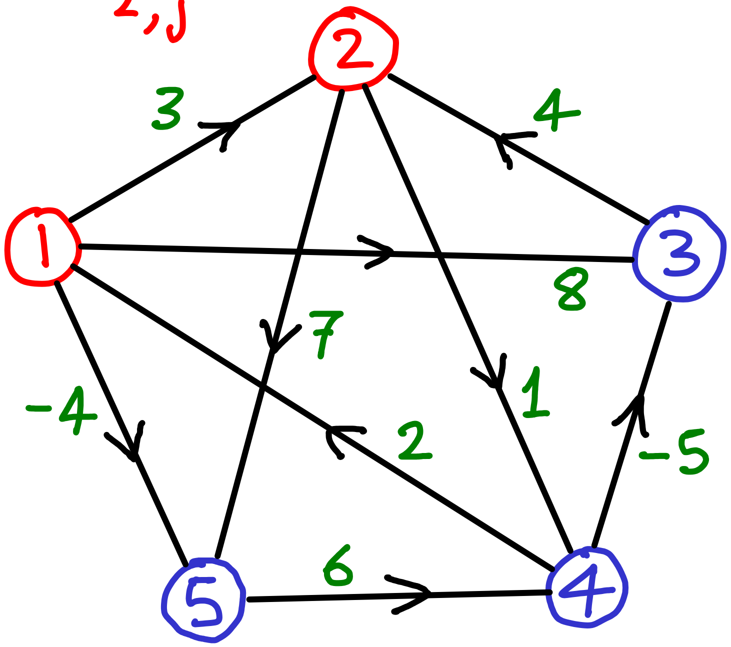
Now let's allow V_2 to help

D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$



$$d^2_{i,j} : \begin{cases} \text{if we don't use } V_2 \text{ then solution is in } D^1 \Rightarrow d^1_{i,j} \\ \text{else} \quad i \rightsquigarrow V_2 \rightsquigarrow j \Rightarrow d^1_{i,2} + d^1_{2,j} \end{cases}$$

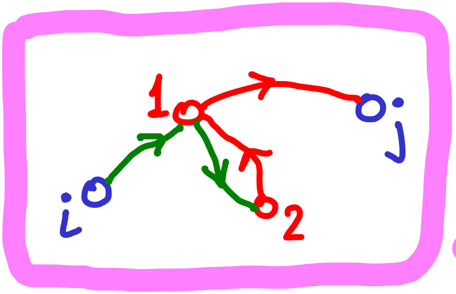
$$D^0 = W \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad D^1 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$



D^1 = all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, via V_1

Now let's allow V_2 to help

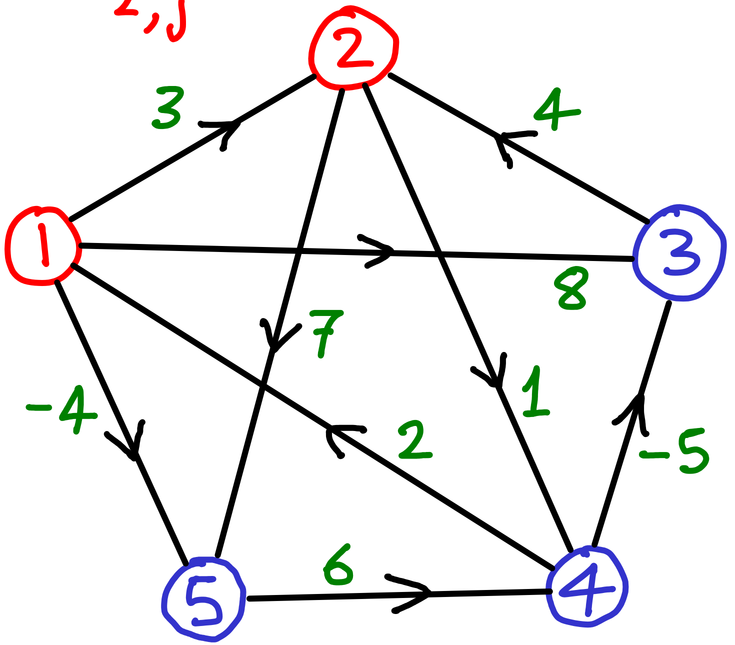
D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$



$$d^2_{i,j} : \begin{cases} \text{if we don't use } V_2 \text{ then solution is in } D^1 \Rightarrow d^1_{i,j} \\ \text{else} \quad i \rightsquigarrow V_2 \rightsquigarrow j \Rightarrow d^1_{i,2} + d^1_{2,j} \end{cases}$$

$$D^0 = W = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

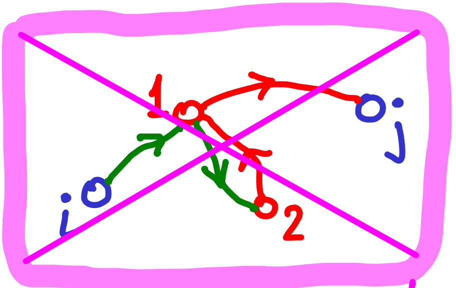
$$D^1 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$



D^1 = all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, via V_1

Now let's allow V_2 to help

D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$

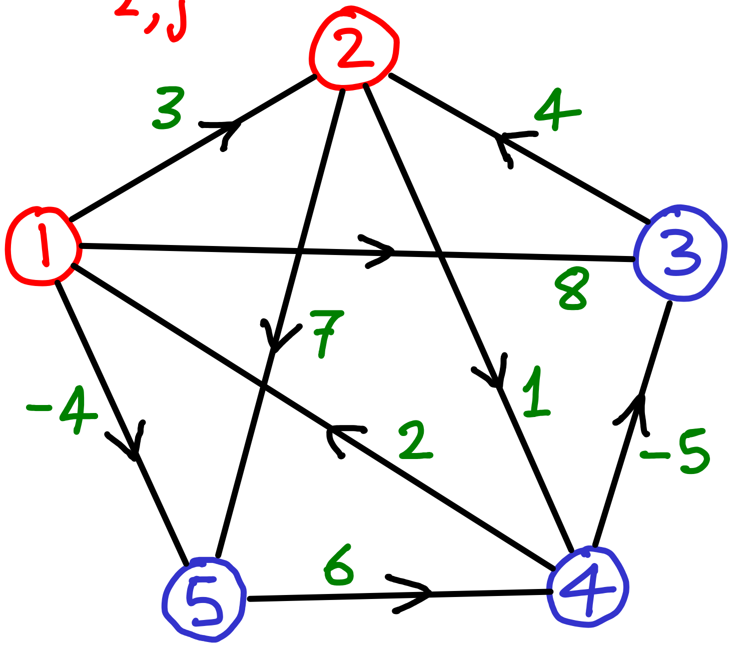


No negative cycles

$$d^2_{i,j} : \begin{cases} \text{if we don't use } V_2 \text{ then solution is in } D^1 \Rightarrow d^1_{i,j} \\ \text{else } \quad i \rightsquigarrow V_2 \rightsquigarrow j \Rightarrow \underline{d^1_{i,2}} + d^1_{2,j} \end{cases}$$

$$D^0 = W = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

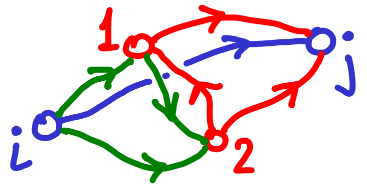
$$D^1 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$



D^1 = all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, via V_1

Now let's allow V_2 to help

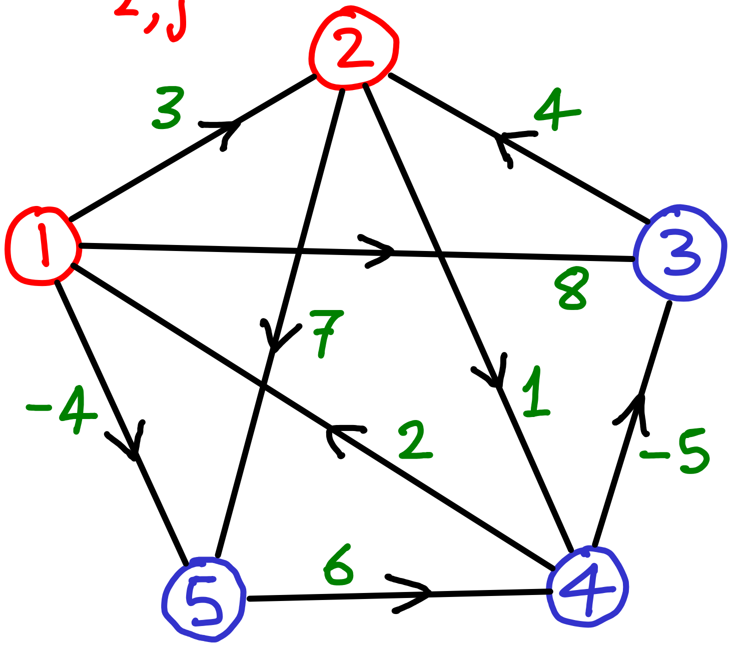
D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$



$$d^2_{i,j} : \begin{cases} \text{if we don't use } V_2 \text{ then solution is in } D^1 \Rightarrow d^1_{i,j} \\ \text{else } \quad i \rightsquigarrow V_2 \rightsquigarrow j \Rightarrow d^1_{i,2} + d^1_{2,j} \end{cases}$$

$$D^0 = W = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

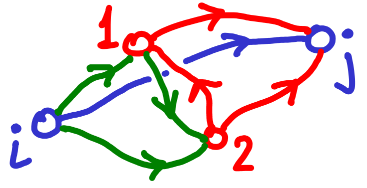
$$D^1 = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$



D^1 = all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, via V_1

Now let's allow V_2 to help

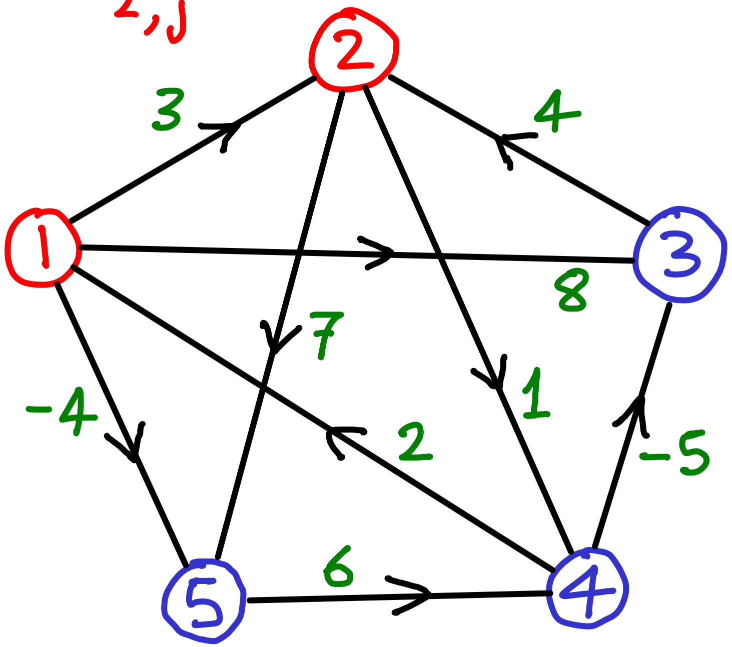
D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$



$$d^2_{i,j} : \begin{cases} \text{if we don't use } V_2 \text{ then solution is in } D^1 \Rightarrow d^1_{i,j} \\ \text{else} \quad i \rightsquigarrow V_2 \rightsquigarrow j \Rightarrow d^1_{i,2} + d^1_{2,j} \end{cases}$$

$$D^1 = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

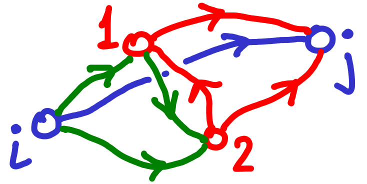
$$D^2 = \begin{bmatrix} \; \end{bmatrix}$$



D^1 = all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, via V_1

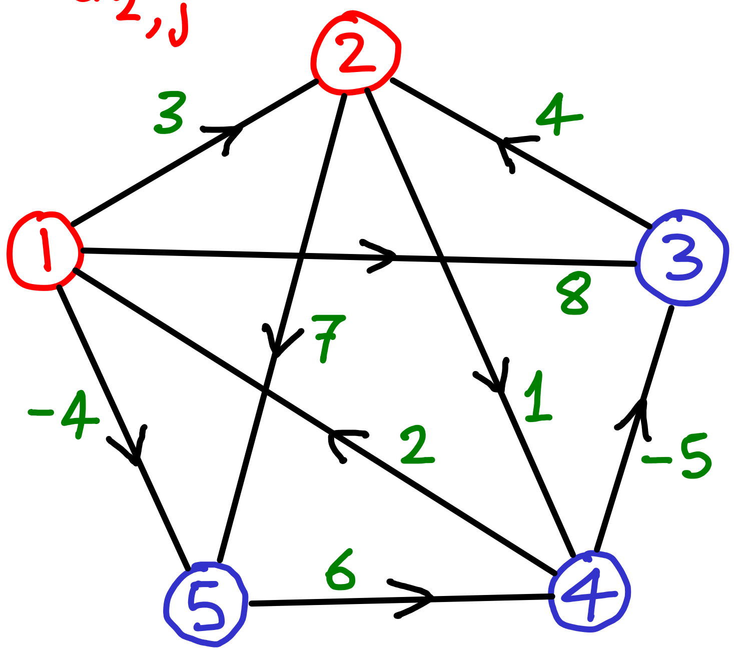
Now let's allow V_2 to help

D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$



$$d^2_{i,j} : \begin{cases} \text{if we don't use } V_2 \text{ then solution is in } D^1 \Rightarrow d^1_{i,j} \\ \text{else } \quad i \rightsquigarrow V_2 \rightsquigarrow j \Rightarrow d^1_{i,2} + d^1_{2,j} \end{cases}$$

$$D^1 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad D^2 \begin{bmatrix} & 3 & & & \\ \infty & 0 & \infty & 1 & 7 \\ & 4 & & & \\ & 5 & & & \\ & \infty & & & \end{bmatrix}$$



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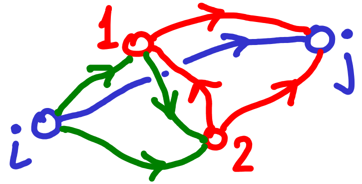
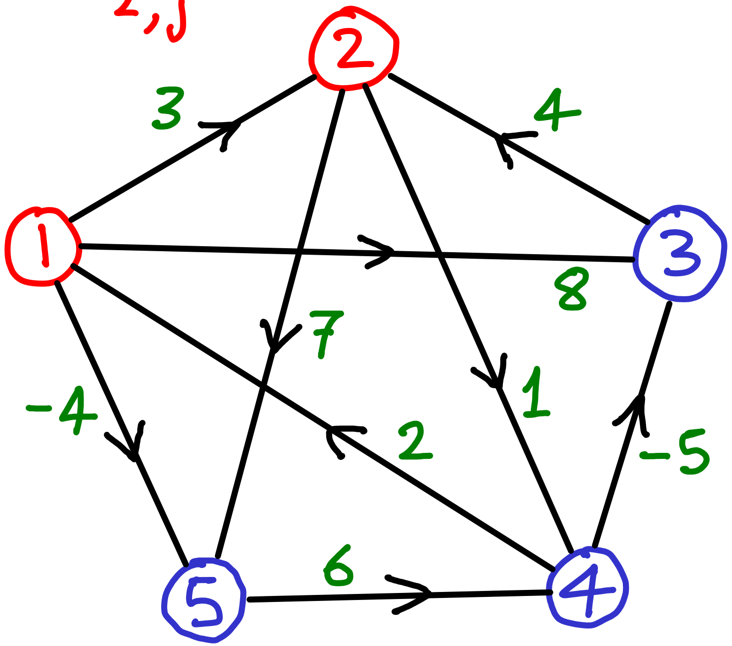
Now let's allow V_2 to help

D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$



$$d^2_{i,j} : \begin{cases} \text{if we don't use } V_2 \text{ then solution is in } D^1 \Rightarrow d^1_{i,j} \\ \text{else } \quad i \rightsquigarrow V_2 \rightsquigarrow j \Rightarrow d^1_{i,2} + d^1_{2,j} \end{cases}$$

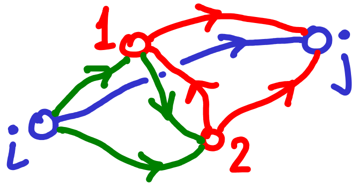
$$D^1 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad D^2 \begin{bmatrix} & 3 & ? & & \\ \infty & 0 & \infty & 1 & 7 \\ & 4 & & & \\ & 5 & & & \\ & \infty & & & \end{bmatrix}$$



D^1 = all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, via V_1

Now let's allow V_2 to help

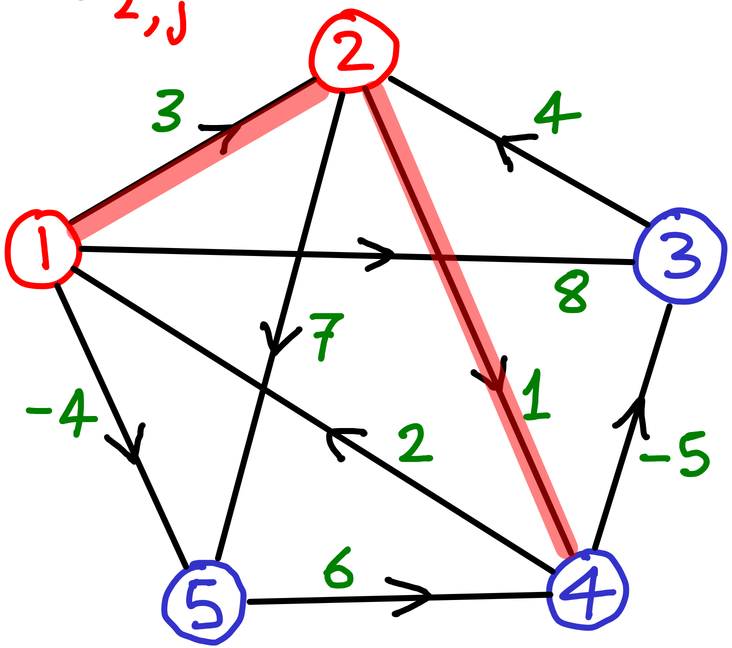
D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$



$$d^2_{i,j} : \begin{cases} \text{if we don't use } V_2 \text{ then solution is in } D^1 \Rightarrow d^1_{i,j} \\ \text{else} \quad i \rightsquigarrow V_2 \rightsquigarrow j \Rightarrow d^1_{i,2} + d^1_{2,j} \end{cases}$$

$$D^1 \quad \begin{array}{cc} \begin{array}{ccccc} & d^1_{1,2} & & & \end{array} & \\ \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} & \begin{array}{c} \\ d^1_{2,4} \\ \\ \\ \\ \end{array} \end{array}$$

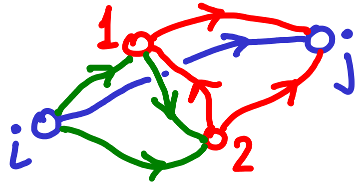
$$D^2 \quad \begin{array}{c} \begin{array}{ccccc} & & & d^2_{1,4} & \end{array} \\ \begin{bmatrix} & 3 & & ? & \\ \infty & 0 & \infty & 1 & 7 \\ & 4 & & & \\ & 5 & & & \\ & \infty & & & \end{bmatrix} \end{array}$$



D^1 = all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, via V_1

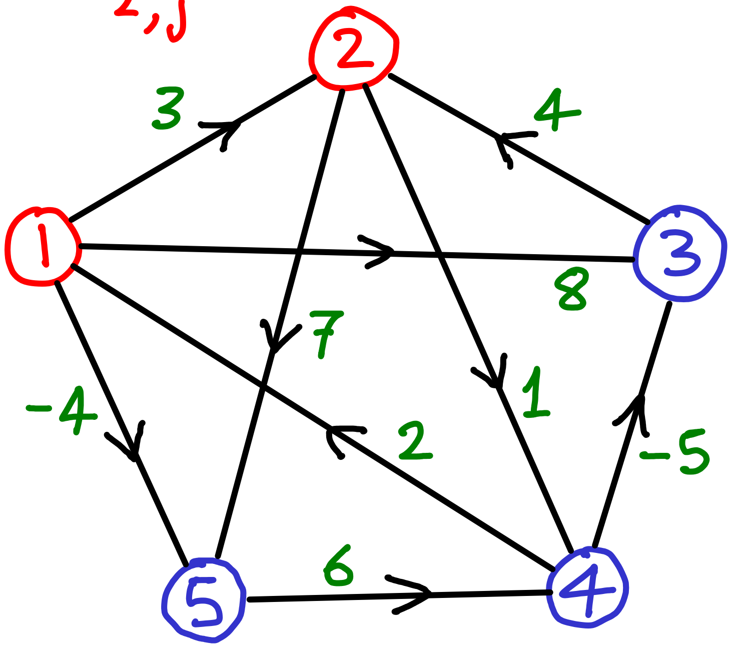
Now let's allow V_2 to help

D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$



$$d^2_{i,j} : \begin{cases} \text{if we don't use } V_2 \text{ then solution is in } D^1 \Rightarrow d^1_{i,j} \\ \text{else} \quad i \rightsquigarrow V_2 \rightsquigarrow j \Rightarrow d^1_{i,2} + d^1_{2,j} \end{cases}$$

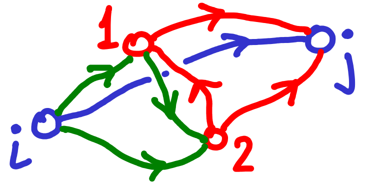
$$D^1 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad D^2 \begin{bmatrix} & 3 & & 4 & \\ \infty & 0 & \infty & 1 & 7 \\ & 4 & & & \\ & 5 & & & \\ & \infty & & & \end{bmatrix}$$



D^1 = all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, via V_1

Now let's allow V_2 to help

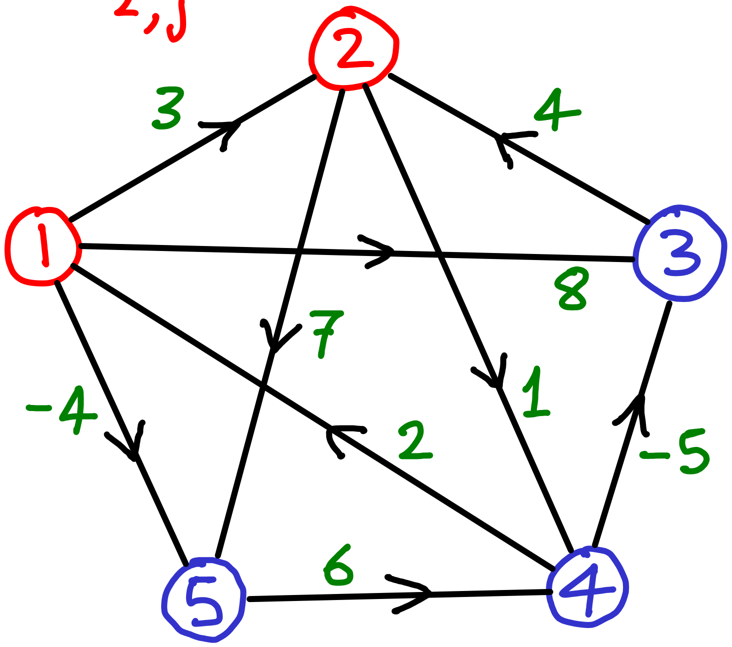
D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$



$$d^2_{i,j} : \begin{cases} \text{if we don't use } V_2 \text{ then solution is in } D^1 \Rightarrow d^1_{i,j} \\ \text{else } \quad i \rightsquigarrow V_2 \rightsquigarrow j \Rightarrow d^1_{i,2} + d^1_{2,j} \end{cases}$$

$$D^1 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

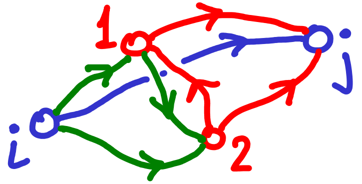
$$D^2 \begin{bmatrix} & 3 & & 4 & \\ \infty & 0 & \infty & 1 & 7 \\ & 4 & & & ? \\ & 5 & & & \\ & \infty & & & \end{bmatrix} d^2_{3,5}$$



D^1 = all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, via V_1

Now let's allow V_2 to help

D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$

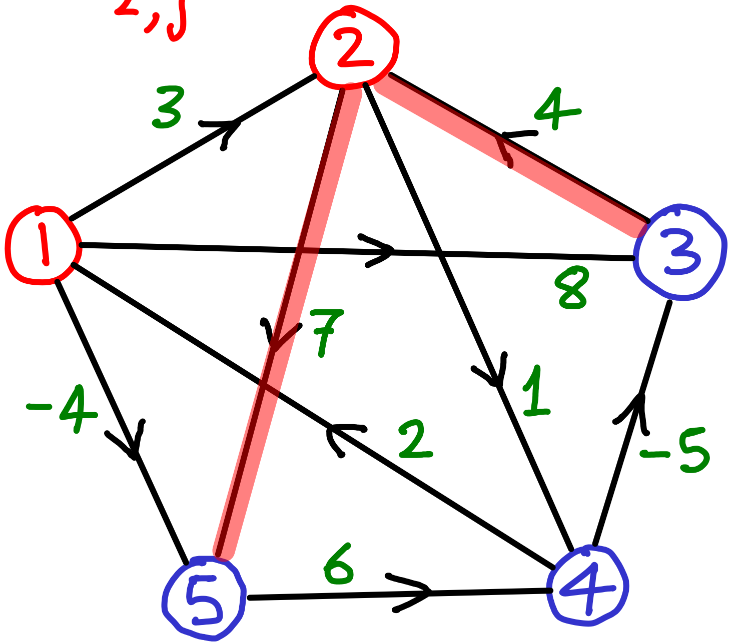


$$d^2_{i,j} : \begin{cases} \text{if we don't use } V_2 \text{ then solution is in } D^1 \Rightarrow d^1_{i,j} \\ \text{else } \quad i \rightsquigarrow V_2 \rightsquigarrow j \quad \Rightarrow d^1_{i,2} + d^1_{2,j} \end{cases}$$

$d^1_{3,2}$

$$D^1 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \; d^1_{2,5}$$

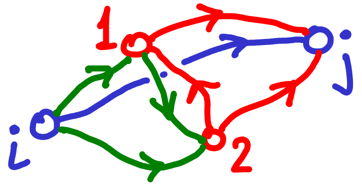
$$D^2 \begin{bmatrix} & 3 & & 4 & \\ \infty & 0 & \infty & 1 & 7 \\ & 4 & & & 11 \\ & 5 & & & \\ & \infty & & & \end{bmatrix} \; d^2_{3,5}$$



D^1 = all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, via V_1

Now let's allow V_2 to help

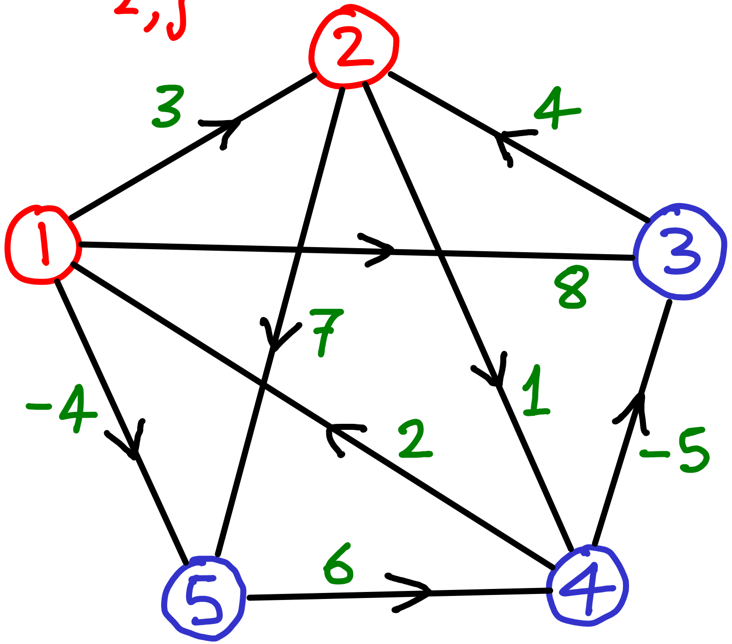
D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$



$$d^2_{i,j} : \begin{cases} \text{if we don't use } V_2 \text{ then solution is in } D^1 \Rightarrow d^1_{i,j} \\ \text{else } \quad i \rightsquigarrow V_2 \rightsquigarrow j \Rightarrow d^1_{i,2} + d^1_{2,j} \end{cases}$$

$$D^1 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

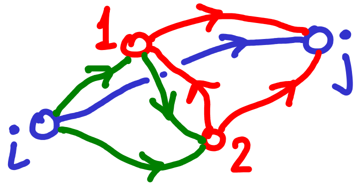
$$D^2 \quad \begin{array}{c} \\ \\ \\ d^2_{4,1} \\ \\ \end{array} \begin{bmatrix} & 3 & & 4 & \\ \infty & 0 & \infty & 1 & 7 \\ & 4 & & & 11 \\ ? & 5 & & & \\ & \infty & & & \end{bmatrix}$$



D^1 = all shortest paths $i \rightsquigarrow j$ with at most 1 intermediate stop, via V_1

Now let's allow V_2 to help

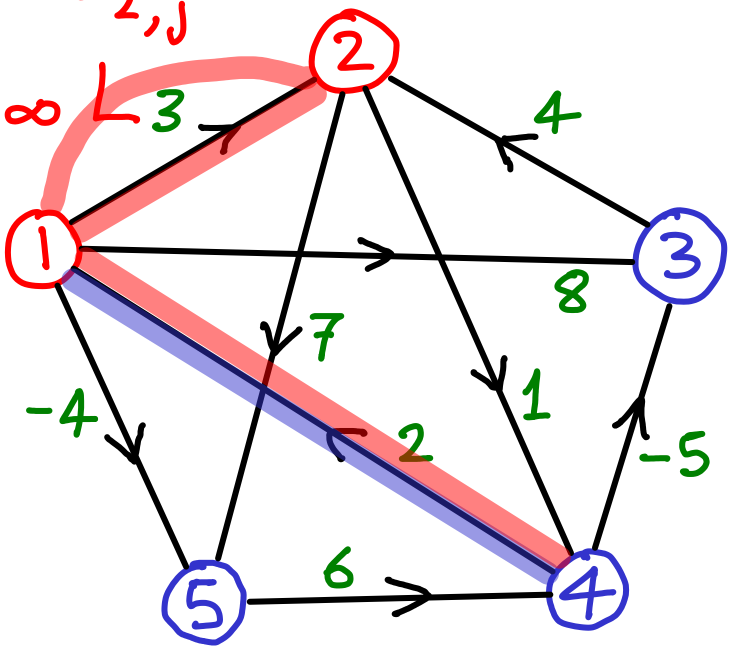
D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$



$$d^2_{i,j} : \begin{cases} \text{if we don't use } V_2 \text{ then solution is in } D^1 \Rightarrow d^1_{i,j} \\ \text{else } \quad i \rightsquigarrow V_2 \rightsquigarrow j \quad \Rightarrow d^1_{i,2} + d^1_{2,j} \end{cases}$$

$$D^1 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

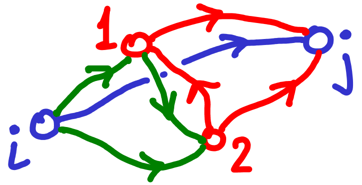
$$D^2 \begin{bmatrix} & 3 & & 4 & \\ \infty & 0 & \infty & 1 & 7 \\ & 4 & & & 11 \\ 2 & 5 & & & \\ & \infty & & & \end{bmatrix}$$



D^1 = all shortest paths $i \leadsto j$ with at most 1 intermediate stop, via V_1

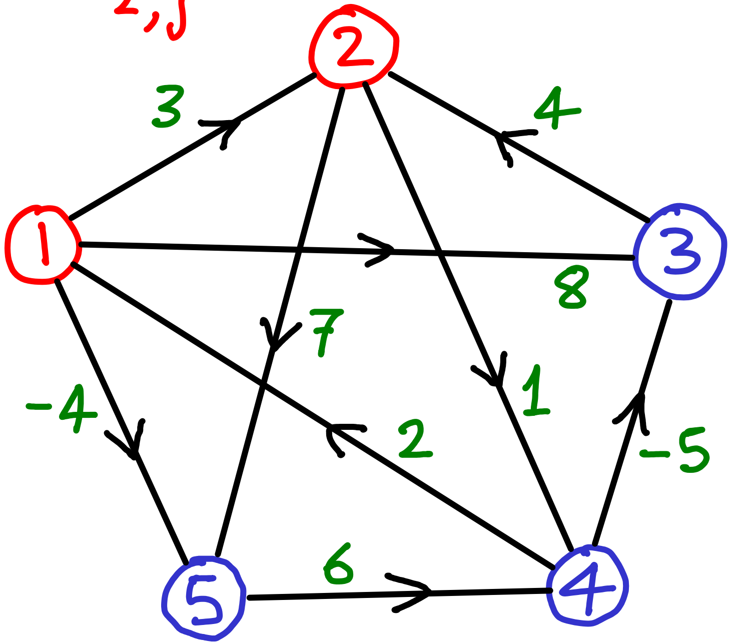
Now let's allow V_2 to help

D^2 = all shortest paths $i \leadsto j$ with at most 2 intermediate stops via $\{V_1, V_2\}$



$$d^2_{i,j} : \begin{cases} \text{if we don't use } V_2 \text{ then solution is in } D^1 \Rightarrow d^1_{i,j} \\ \text{else} \quad i \leadsto V_2 \leadsto j \Rightarrow d^1_{i,2} + d^1_{2,j} \end{cases}$$

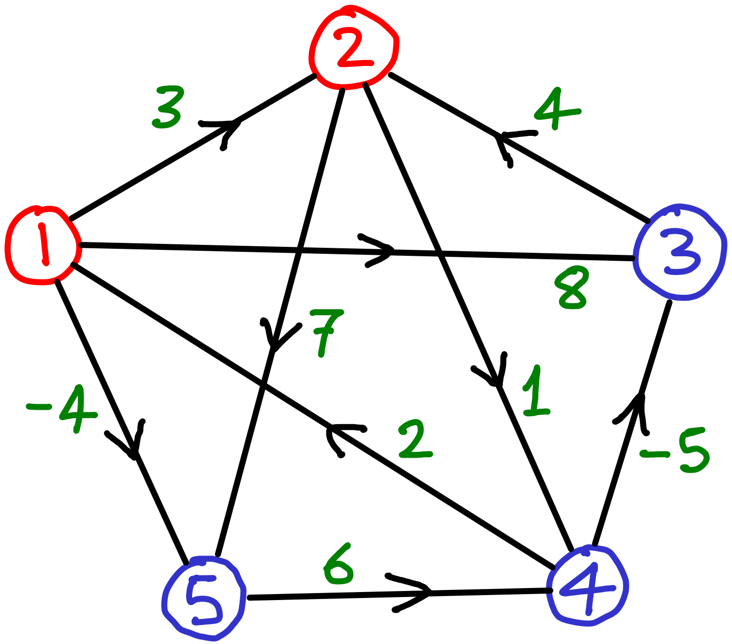
$$D^1 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad D^2 \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$



D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{v_1, v_2\}$

$$D^1 = \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

$$D^2 = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

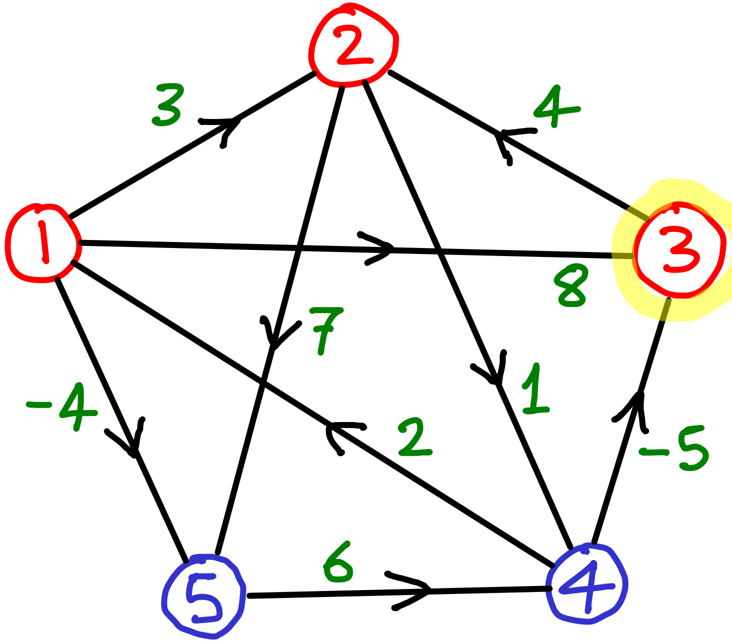


D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$

Now let's allow V_3 to help

D^3 = all shortest paths $i \rightsquigarrow j$ with ≤ 3 intermediate stops via $\{V_1, V_2, V_3\}$

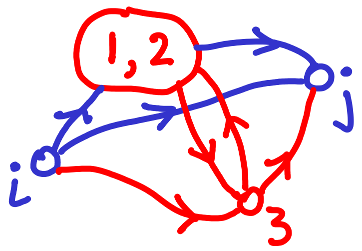
$$D^1 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad D^2 \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$



D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$

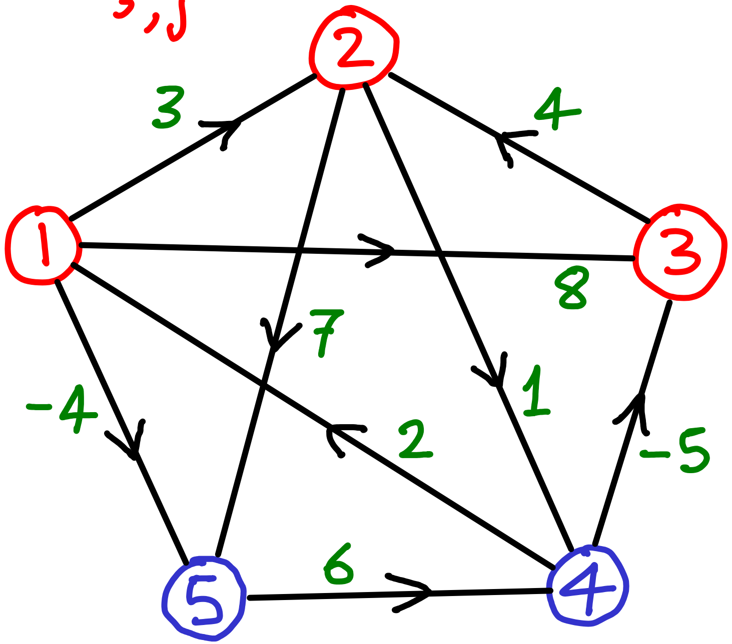
Now let's allow V_3 to help

D^3 = all shortest paths $i \rightsquigarrow j$ with ≤ 3 intermediate stops via $\{V_1, V_2, V_3\}$



$$d^3_{i,j} : \begin{cases} \text{if we don't use } V_3 \text{ then solution is in } D^2 \Rightarrow d^2_{i,j} \\ \text{else} \quad i \rightsquigarrow V_3 \rightsquigarrow j \Rightarrow d^2_{i,3} + d^2_{3,j} \end{cases}$$

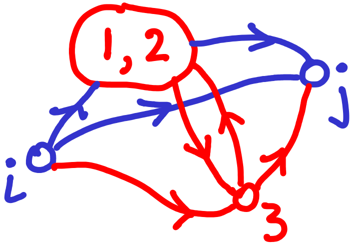
$$D^1 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad D^2 \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$



D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$

Now let's allow V_3 to help

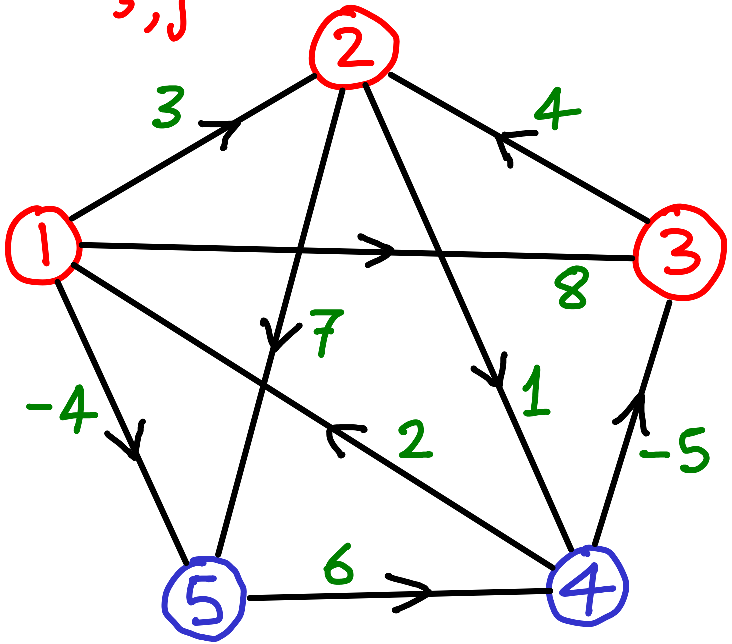
D^3 = all shortest paths $i \rightsquigarrow j$ with ≤ 3 intermediate stops via $\{V_1, V_2, V_3\}$



$$d^3_{i,j} : \begin{cases} \text{if we don't use } V_3 \text{ then solution is in } D^2 \Rightarrow d^2_{i,j} \\ \text{else} \quad i \rightsquigarrow V_3 \rightsquigarrow j \Rightarrow d^2_{i,3} + d^2_{3,j} \end{cases}$$

$$D^1 \begin{bmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

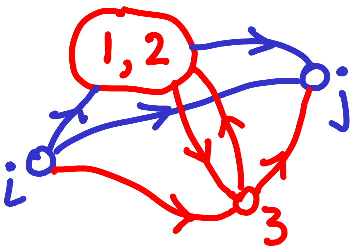
$$D^2 \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$



D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$

Now let's allow V_3 to help

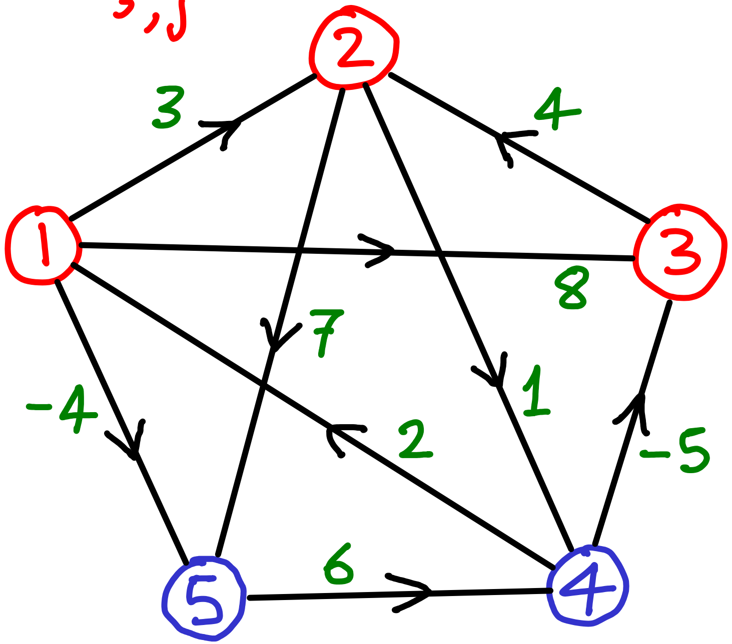
D^3 = all shortest paths $i \rightsquigarrow j$ with ≤ 3 intermediate stops via $\{V_1, V_2, V_3\}$



$$d^3_{i,j} : \begin{cases} \text{if we don't use } V_3 \text{ then solution is in } D^2 \Rightarrow d^2_{i,j} \\ \text{else} \quad i \rightsquigarrow V_3 \rightsquigarrow j \Rightarrow d^2_{i,3} + d^2_{3,j} \end{cases}$$

$$D^2 \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

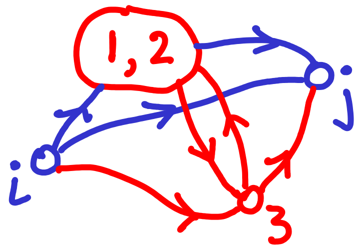
D^3



D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$

Now let's allow V_3 to help

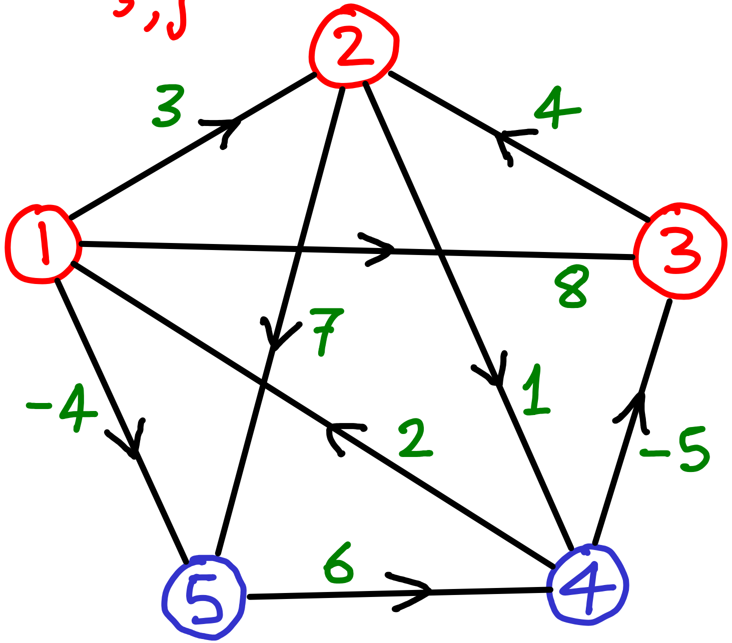
D^3 = all shortest paths $i \rightsquigarrow j$ with ≤ 3 intermediate stops via $\{V_1, V_2, V_3\}$



$$d^3_{i,j} : \begin{cases} \text{if we don't use } V_3 \text{ then solution is in } D^2 \Rightarrow d^2_{i,j} \\ \text{else} \quad i \rightsquigarrow V_3 \rightsquigarrow j \Rightarrow d^2_{i,3} + d^2_{3,j} \end{cases}$$

$$D^2 = \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

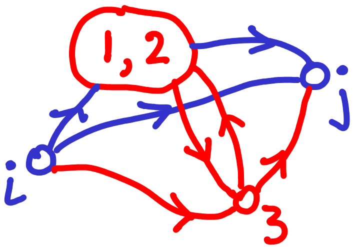
$$D^3 = \begin{bmatrix} & & 8 & & \\ & & \infty & & \\ \infty & 4 & 0 & 5 & 11 \\ & & -5 & & \\ & & \infty & & \end{bmatrix}$$



D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$

Now let's allow V_3 to help

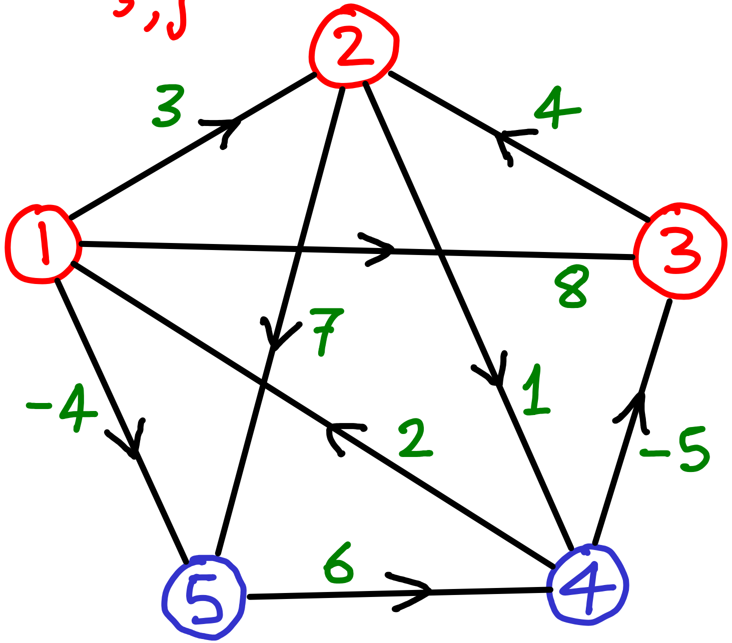
D^3 = all shortest paths $i \rightsquigarrow j$ with ≤ 3 intermediate stops via $\{V_1, V_2, V_3\}$



$$d^3_{i,j} : \begin{cases} \text{if we don't use } V_3 \text{ then solution is in } D^2 \Rightarrow d^2_{i,j} \\ \text{else} \quad i \rightsquigarrow V_3 \rightsquigarrow j \Rightarrow d^2_{i,3} + d^2_{3,j} \end{cases}$$

$$D^2 \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

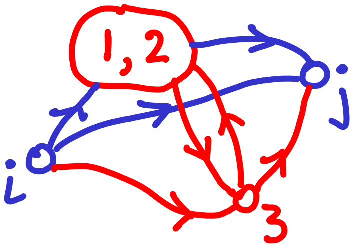
$$D^3 \begin{bmatrix} & & 8 & & \\ & & \infty & & \\ \infty & 4 & 0 & 5 & 11 \\ & ? & -5 & & \\ & & \infty & & \end{bmatrix}$$



D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$

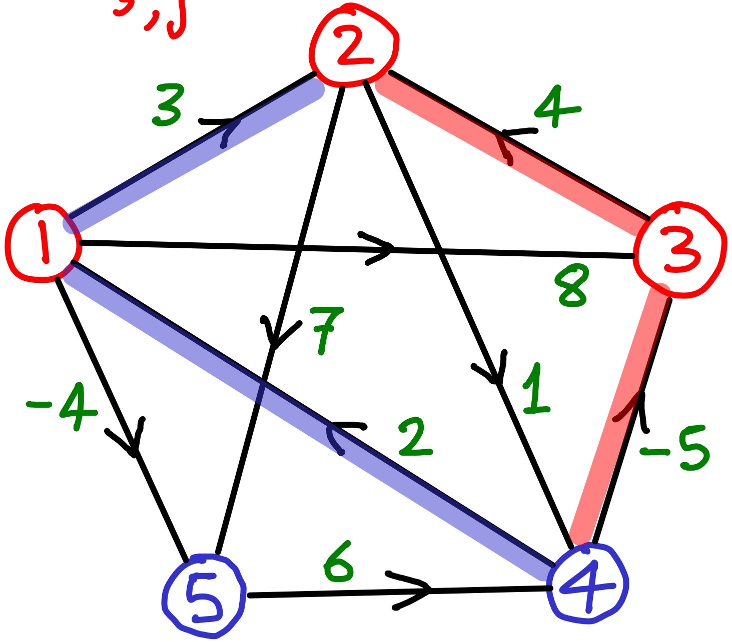
Now let's allow V_3 to help

D^3 = all shortest paths $i \rightsquigarrow j$ with ≤ 3 intermediate stops via $\{V_1, V_2, V_3\}$



$$d^3_{i,j} : \begin{cases} \text{if we don't use } V_3 \text{ then solution is in } D^2 \Rightarrow d^2_{i,j} \\ \text{else} \quad i \rightsquigarrow V_3 \rightsquigarrow j \Rightarrow d^2_{i,3} + d^2_{3,j} \end{cases}$$

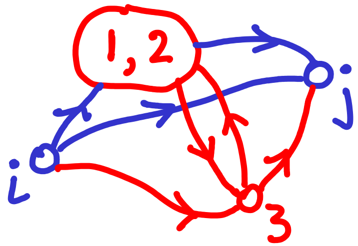
$$D^2 \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad D^3 \begin{bmatrix} & & 8 & & \\ & & \infty & & \\ \infty & 4 & 0 & 5 & 11 \\ & -1 & -5 & & \\ & & \infty & & \end{bmatrix}$$



D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$

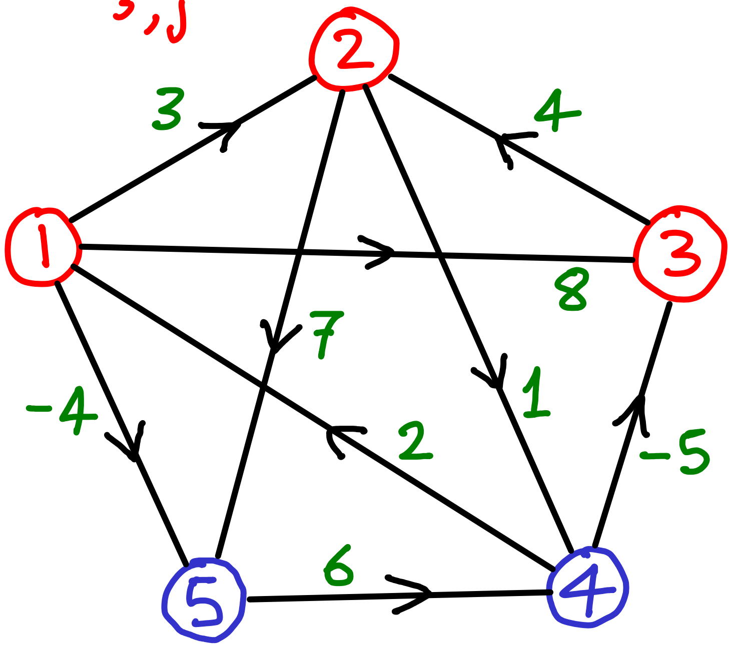
Now let's allow V_3 to help

D^3 = all shortest paths $i \rightsquigarrow j$ with ≤ 3 intermediate stops via $\{V_1, V_2, V_3\}$



$d^3_{i,j} : \begin{cases} \text{if we don't use } V_3 \text{ then solution is in } D^2 \Rightarrow d^2_{i,j} \\ \text{else } \quad i \rightsquigarrow V_3 \rightsquigarrow j \Rightarrow d^2_{i,3} + d^2_{3,j} \end{cases}$

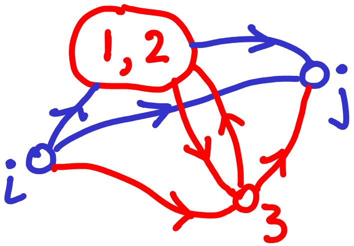
$$D^2 \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad D^3 \begin{bmatrix} & & 8 & & \\ & & \infty & & \\ \infty & 4 & 0 & 5 & 11 \\ & -1 & -5 & & ? \\ & & \infty & & \end{bmatrix}$$



D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$

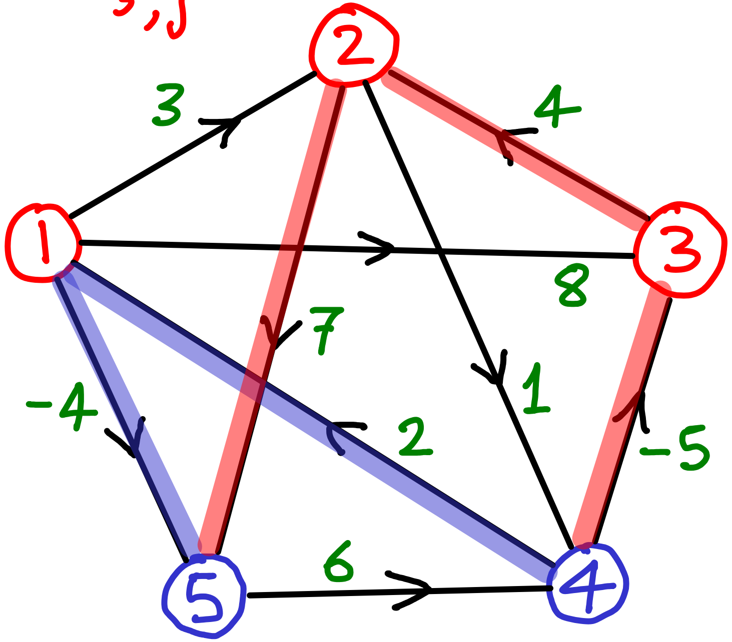
Now let's allow V_3 to help

D^3 = all shortest paths $i \rightsquigarrow j$ with ≤ 3 intermediate stops via $\{V_1, V_2, V_3\}$



$$d^3_{i,j} : \begin{cases} \text{if we don't use } V_3 \text{ then solution is in } D^2 \Rightarrow d^2_{i,j} \\ \text{else} \quad i \rightsquigarrow V_3 \rightsquigarrow j \Rightarrow d^2_{i,3} + d^2_{3,j} \end{cases}$$

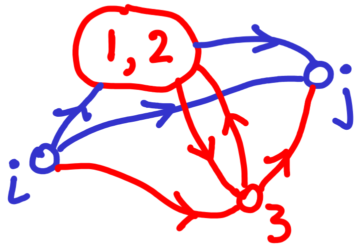
$$D^2 \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad D^3 \begin{bmatrix} & & 8 & & \\ & & \infty & & \\ \infty & 4 & 0 & 5 & 11 \\ & -1 & -5 & & -2 \\ & & \infty & & \end{bmatrix}$$



D^2 = all shortest paths $i \rightsquigarrow j$ with at most 2 intermediate stops via $\{V_1, V_2\}$

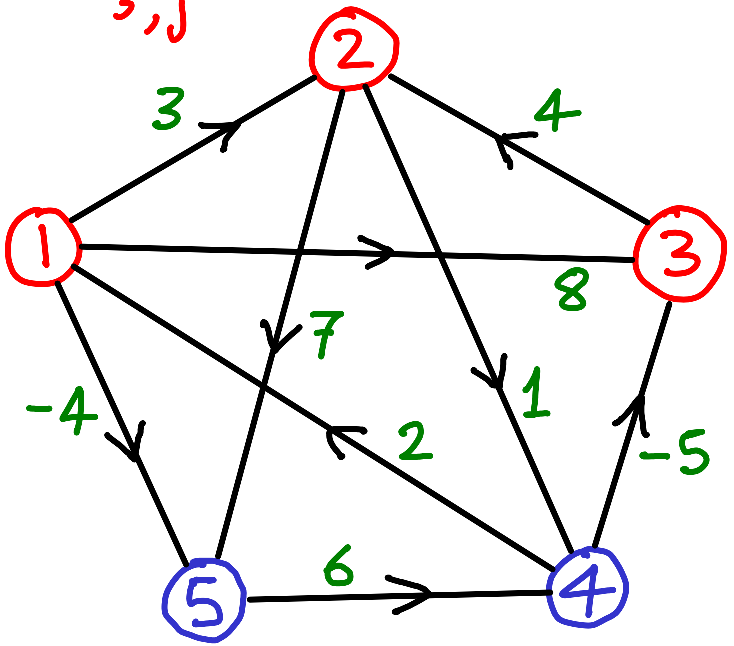
Now let's allow V_3 to help

D^3 = all shortest paths $i \rightsquigarrow j$ with ≤ 3 intermediate stops via $\{V_1, V_2, V_3\}$



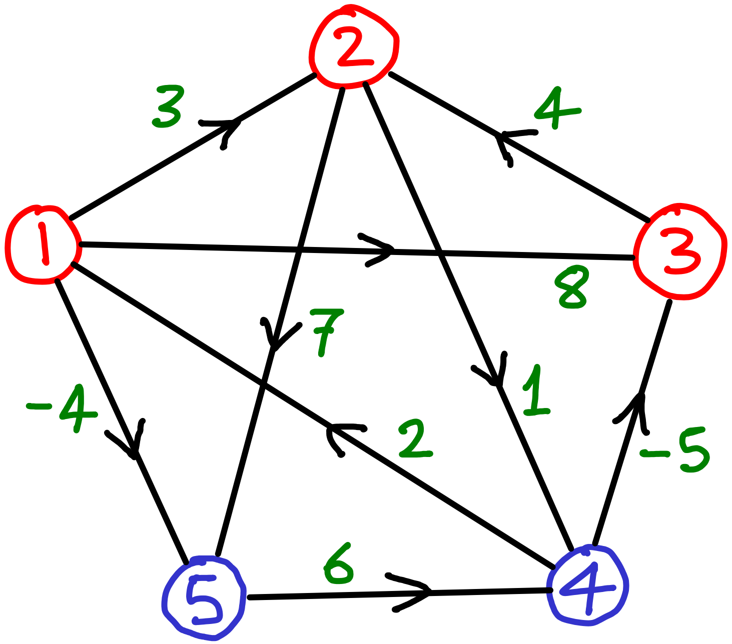
$$d^3_{i,j} : \begin{cases} \text{if we don't use } V_3 \text{ then solution is in } D^2 \Rightarrow d^2_{i,j} \\ \text{else } \quad i \rightsquigarrow V_3 \rightsquigarrow j \Rightarrow d^2_{i,3} + d^2_{3,j} \end{cases}$$

$$D^2 \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad D^3 \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$



D^k = all shortest paths $i \rightsquigarrow j$ with $\leq k$ intermediate stops via $\{v_1 \ldots v_k\}$

$$D^k \; (k=3) \quad \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

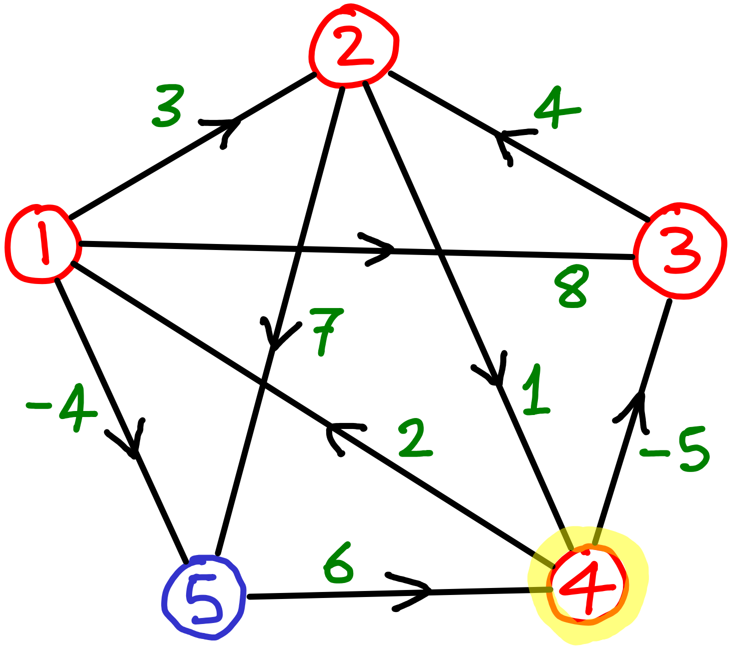


D^k = all shortest paths $i \rightsquigarrow j$ with $\leq k$ intermediate stops via $\{v_1 \dots v_k\}$

Next iteration: allow v_{k+1} to help

D^{k+1} = all shortest paths $i \rightsquigarrow j$ with $\leq k+1$ intermediate stops via $\{v_1 \dots v_{k+1}\}$

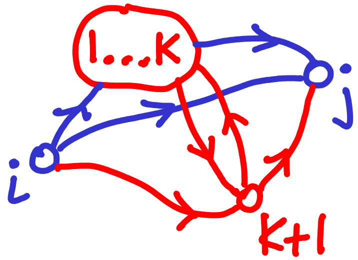
$$D^k \; (k=3) \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix} \qquad D^{k+1} \; (D^4) \begin{bmatrix} \; & \; & \; & \; & \; \end{bmatrix}$$



D^k = all shortest paths $i \rightsquigarrow j$ with $\leq K$ intermediate stops via $\{V_1 \dots V_K\}$

Next iteration: allow V_{K+1} to help

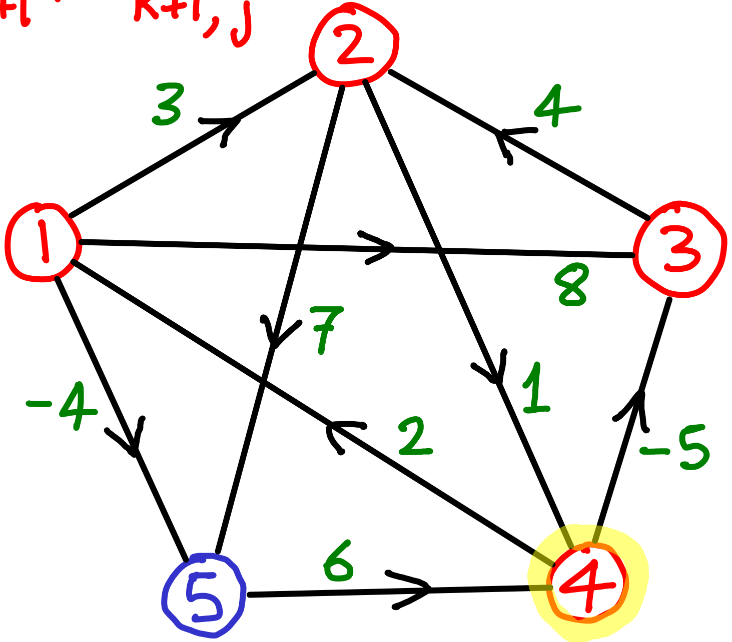
D^{K+1} = all shortest paths $i \rightsquigarrow j$ with $\leq k+1$ intermediate stops via $\{V_1 \dots V_{K+1}\}$



$$d_{i,j}^{K+1} : \begin{cases} \text{if we don't use } V_{K+1} \text{ then solution is in } D^K \Rightarrow d_{i,j}^K \\ \text{else } \quad i \rightsquigarrow V_{K+1} \rightsquigarrow j \Rightarrow d_{i,K+1}^K + d_{K+1,j}^K \end{cases}$$

$$D^K \ (K=3) \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

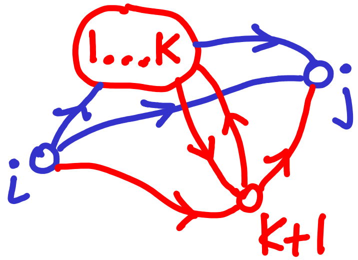
$$D^{K+1} \ (D^4) \begin{bmatrix} \quad & \quad & \quad & \quad & \quad \end{bmatrix}$$



D^k = all shortest paths $i \rightsquigarrow j$ with $\leq k$ intermediate stops via $\{v_1 \dots v_k\}$

Next iteration: allow v_{k+1} to help

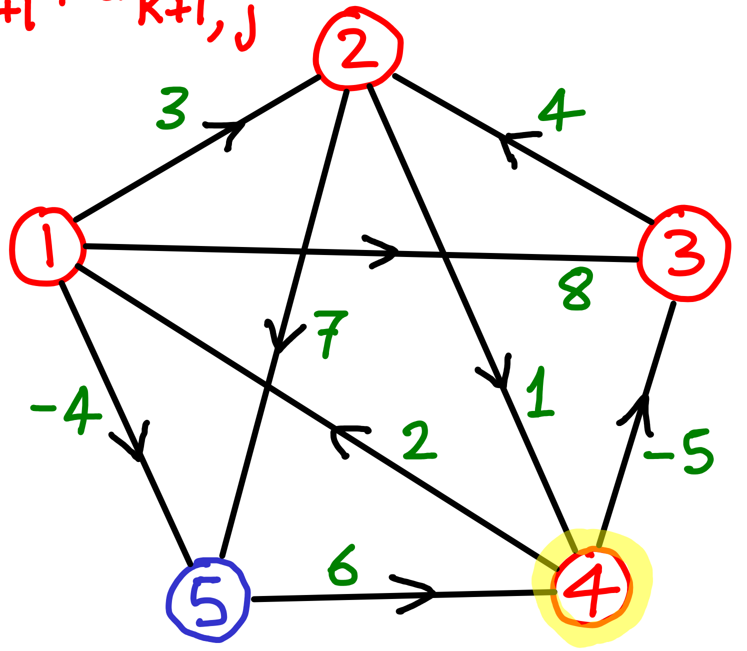
D^{k+1} = all shortest paths $i \rightsquigarrow j$ with $\leq k+1$ intermediate stops via $\{v_1 \dots v_{k+1}\}$



$$d^{k+1}_{i,j} : \begin{cases} \text{if we don't use } v_{k+1} \text{ then solution is in } D^k \Rightarrow d^k_{i,j} \\ \text{else } \quad i \rightsquigarrow v_{k+1} \rightsquigarrow j \Rightarrow d^k_{i,k+1} + d^k_{k+1,j} \end{cases}$$

$$D^k \; (k=3) \begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

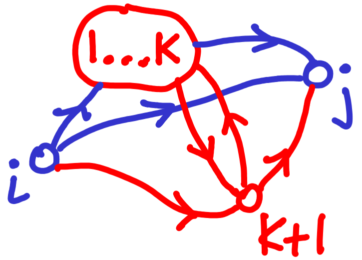
$$D^{k+1} \; (D^4) \begin{bmatrix} & & & 4 & \\ & & & 1 & \\ & & & 5 & \\ 2 & -1 & -5 & 0 & -2 \\ & & & 6 & \end{bmatrix}$$



D^k = all shortest paths $i \rightsquigarrow j$ with $\leq k$ intermediate stops via $\{v_1 \dots v_k\}$

Next iteration: allow v_{k+1} to help

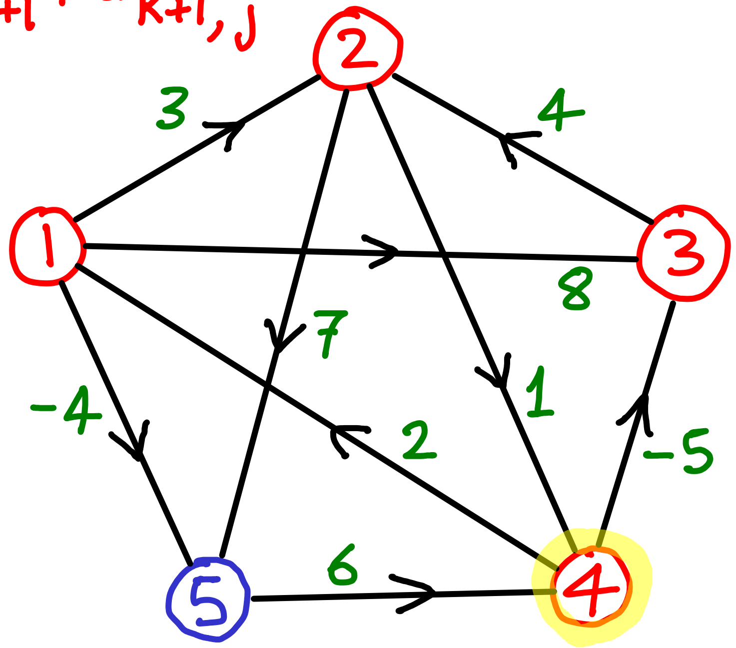
D^{k+1} = all shortest paths $i \rightsquigarrow j$ with $\leq k+1$ intermediate stops via $\{v_1 \dots v_{k+1}\}$



$$d^{k+1}_{i,j} : \begin{cases} \text{if we don't use } v_{k+1} \text{ then solution is in } D^k \Rightarrow d^k_{i,j} \\ \text{else } \quad i \rightsquigarrow v_{k+1} \rightsquigarrow j \Rightarrow d^k_{i,k+1} + d^k_{k+1,j} \end{cases}$$

D^k $(k=3)$
$$\begin{bmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{bmatrix}$$

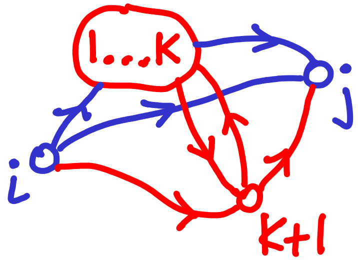
D^{k+1} (D^4)
$$\begin{bmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix}$$



D^k = all shortest paths $i \rightsquigarrow j$ with $\leq K$ intermediate stops via $\{V_1 \ldots V_K\}$

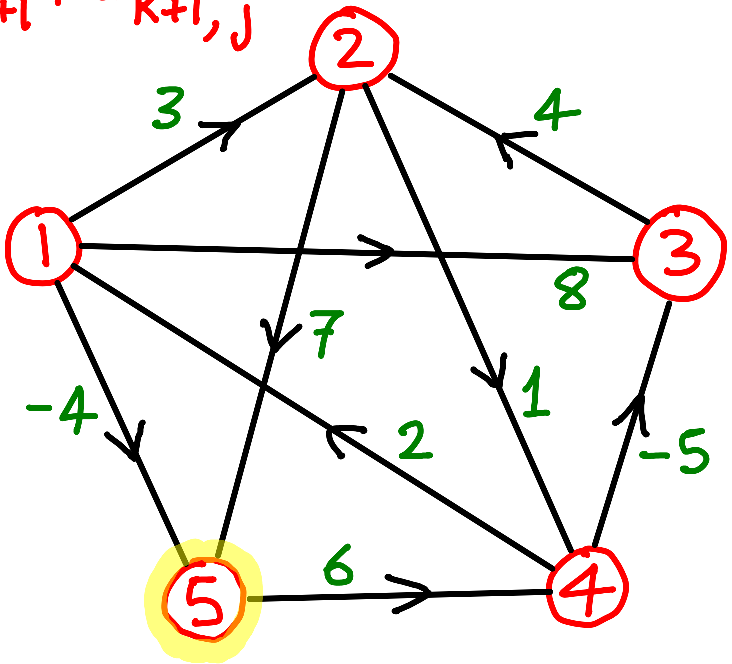
Next iteration: allow V_{K+1} to help

D^{K+1} = all shortest paths $i \rightsquigarrow j$ with $\leq k+1$ intermediate stops via $\{V_1 \ldots V_{K+1}\}$



$$d_{i,j}^{K+1} : \begin{cases} \text{if we don't use } V_{K+1} \text{ then solution is in } D^K \Rightarrow d_{i,j}^K \\ \text{else } \quad i \rightsquigarrow V_{K+1} \rightsquigarrow j \Rightarrow d_{i,K+1}^K + d_{K+1,j}^K \end{cases}$$

$$D^K \ (K=4) \begin{bmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix} \qquad D^{K+1} \ (D^5) \begin{bmatrix} \ \end{bmatrix}$$



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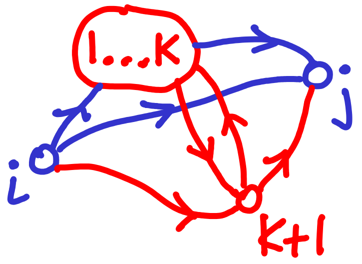

$$d^{K+1}_{i,j} : \begin{cases} \text{if we don't use } V_{K+1} \text{ then solution is in } D^K \Rightarrow d^K_{i,j} \\ \text{else} \quad i \rightsquigarrow V_{K+1} \rightsquigarrow j \Rightarrow d^K_{i,K+1} + d^K_{K+1,j} \end{cases}$$

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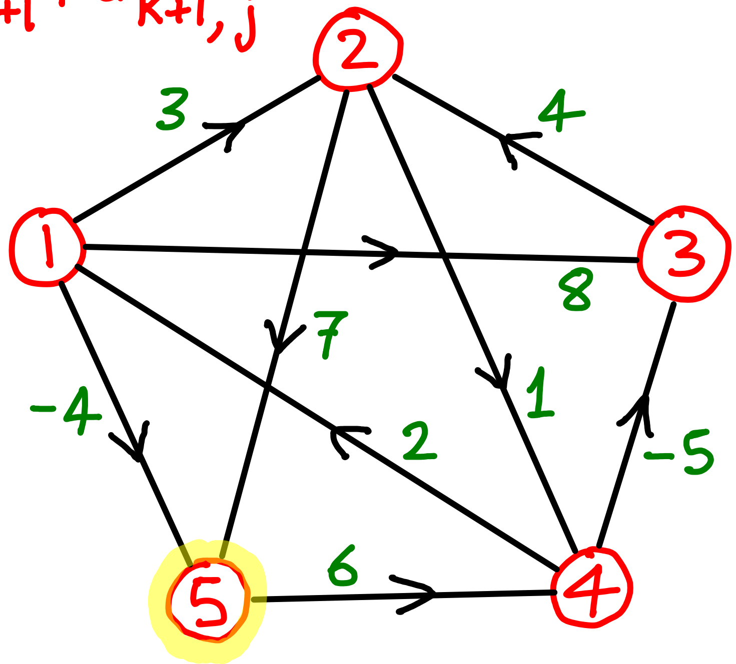

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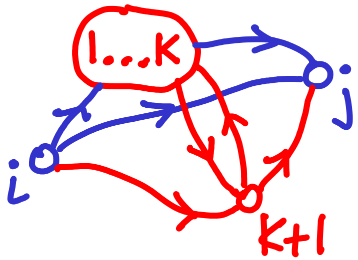

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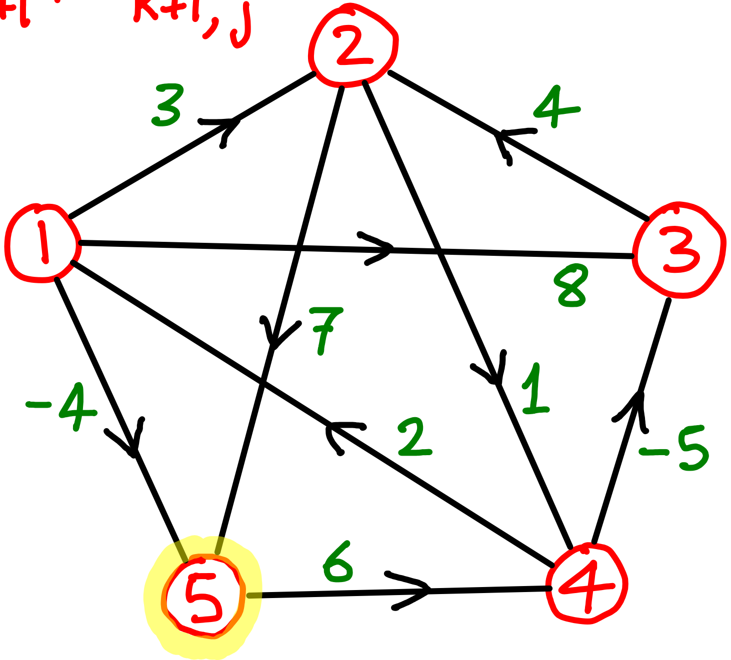
$$D^{K+1} \; (D^5) \begin{array}{c} \begin{array}{ccccc} & & (1,3) & & \end{array} \\ \begin{bmatrix} & & ? & & -4 \\ & & & & -1 \\ & & & & 3 \\ & & & & -2 \\ 8 & 5 & 1 & 6 & 0 \end{bmatrix} \end{array}$$



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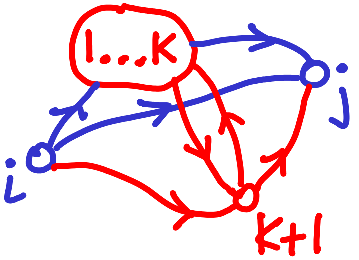

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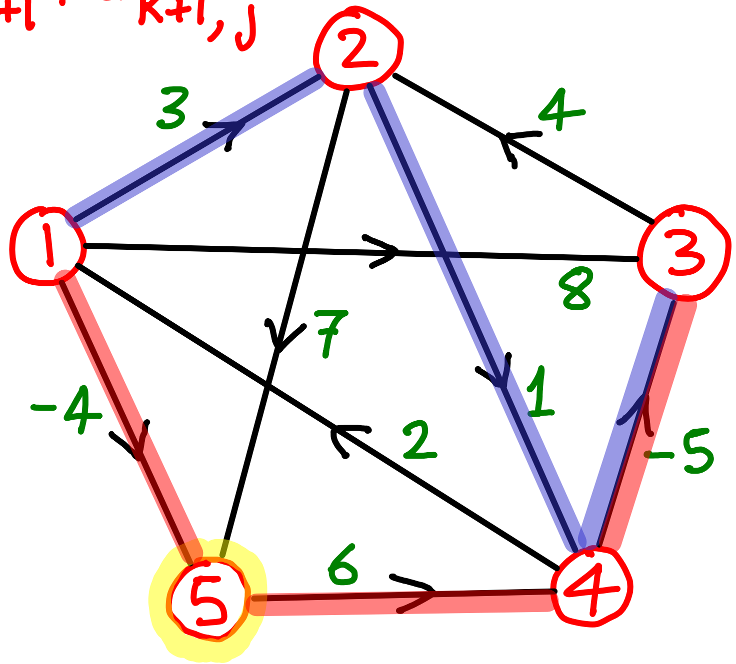

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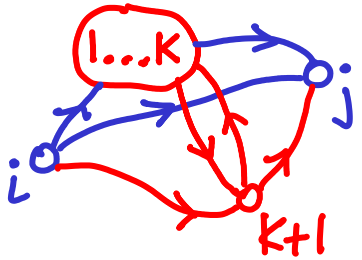

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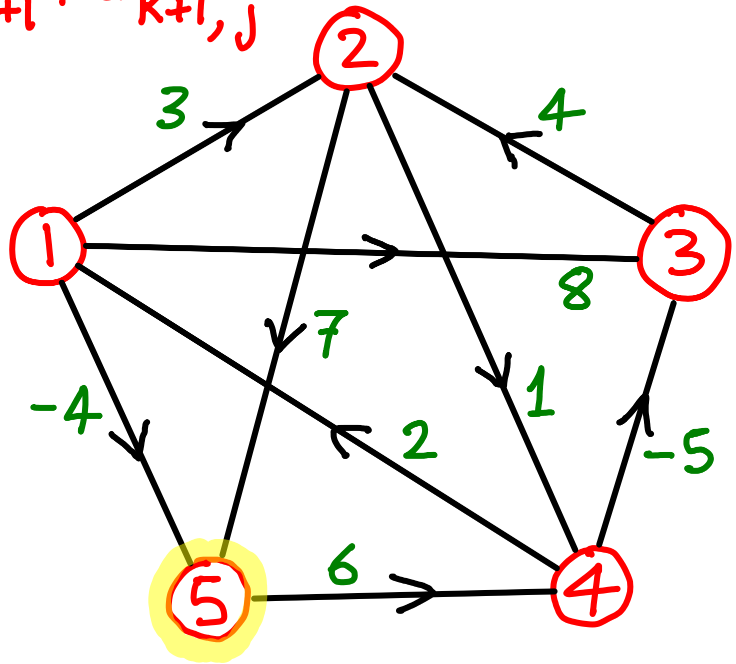

FLOYD-WARSHALL ALGORITHM (Floyd-Roy-Kleene-Warshall)

recap

$$d_{i,j}^{K+1} : \begin{cases} \text{if we don't use } V_{K+1} \text{ then solution is in } D^K \Rightarrow d_{i,j}^K \\ \text{else} \quad i \rightsquigarrow V_{K+1} \rightsquigarrow j \Rightarrow d_{i,K+1}^K + d_{K+1,j}^K \end{cases}$$

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$$d_{i,j}^{K+1} = \min\{ d_{i,j}^{K} , d_{i,K+1}^{K} + d_{K+1,j}^{K} \}$$

FLOYD-WARSHALL ALGORITHM (Floyd-Roy-Kleene-Warshall)

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$$d^{k+1}_{i,j} : \begin{cases} \text{if we don't use } V_{k+1} \text{ then solution is in } D^k \Rightarrow d^k_{i,j} \\ \text{else } \quad i \rightsquigarrow V_{k+1} \rightsquigarrow j \Rightarrow d^k_{i,k+1} + d^k_{k+1,j} \end{cases}$$

$$d^{k+1}_{i,j} = \min\{ d^k_{i,j} \, , \, d^k_{i,k+1} + d^k_{k+1,j} \}$$

Dynamic programming: $\underbrace{V^2 \text{ elements} \times O(1) \text{ per element}}_{V \text{ iterations}}$ $k = 1 \ldots V$

$\Theta(V^3)$

beats repeated squaring

(no negative cycles)

run SSSP, V times

$V \cdot O(V \cdot E)$	w/ B-F
require weights ≥ 0	
$V \cdot O(E \log V)$	w/ bin. heap Dijkstra (adj. list)
$V \cdot \cancel{\Theta(V^2)}$	w/ "array" Dijkstra (adj. matrix)
$V \cdot O(E + V \log V)$	w/ Fibonacci heap Dijkstra (adj. list)

$\Theta(V^3)$ Floyd-Warshall
allows weights < 0

Next:
match the other Dijkstra bounds allowing weights < 0

(still no cycles < 0)

$\Theta(V^3)$ Floyd-Warshall
allows weights < 0

run SSSP, V times

$V \cdot O(V \cdot E)$	w/ B-F
require weights ≥ 0	
$V \cdot O(E \log V)$	w/ bin. heap Dijkstra (adj. list)
$\cancel{V \cdot \Theta(V^2)}$	w/ "array" Dijkstra (adj. matrix)
$V \cdot O(E + V \log V)$	w/ Fibonacci heap Dijkstra (adj. list)

JOHNSON'S ALGORITHM

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$\approx$ 1 Bellman-Ford + ADJUST WEIGHTS + $V \cdot$ Dijkstra

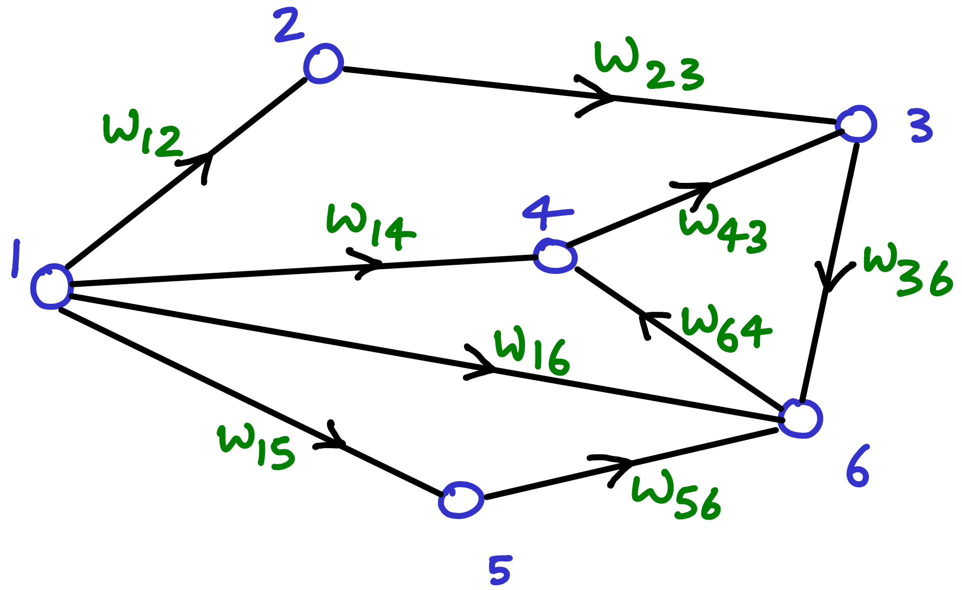
JOHNSON'S ALGORITHM

$\approx$ 1 Bellman-Ford + ADJUST WEIGHTS + V · Dijkstra

- need weights ≥ 0
- need same shortest path structure

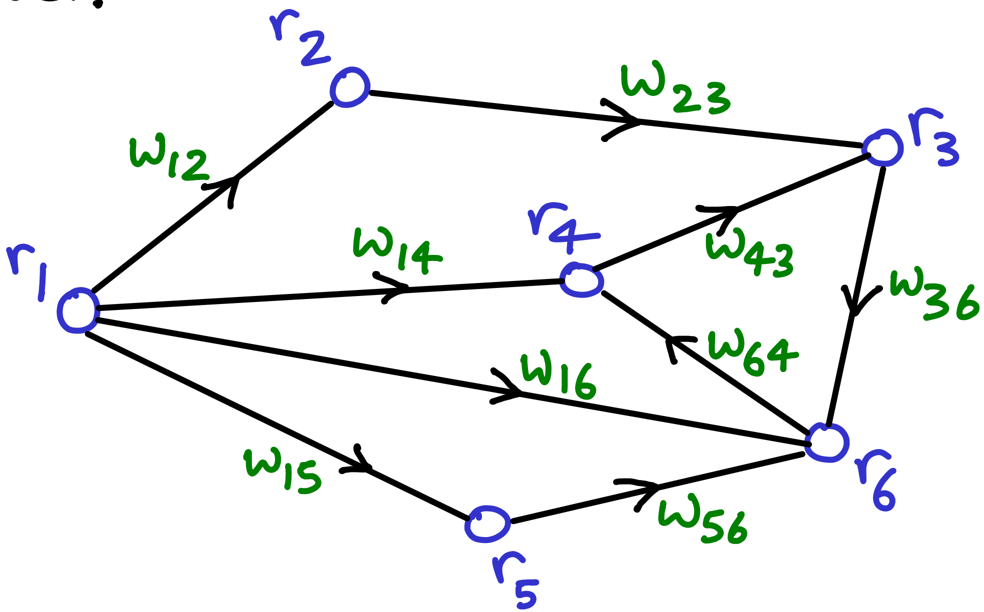
assume adj. list

Input weights: w_{ij}



For every vertex k $(1 \leq k \leq V)$
let r_k be some real number.

Input weights: w_{ij}



For every vertex k $(1 \le k \le V)$
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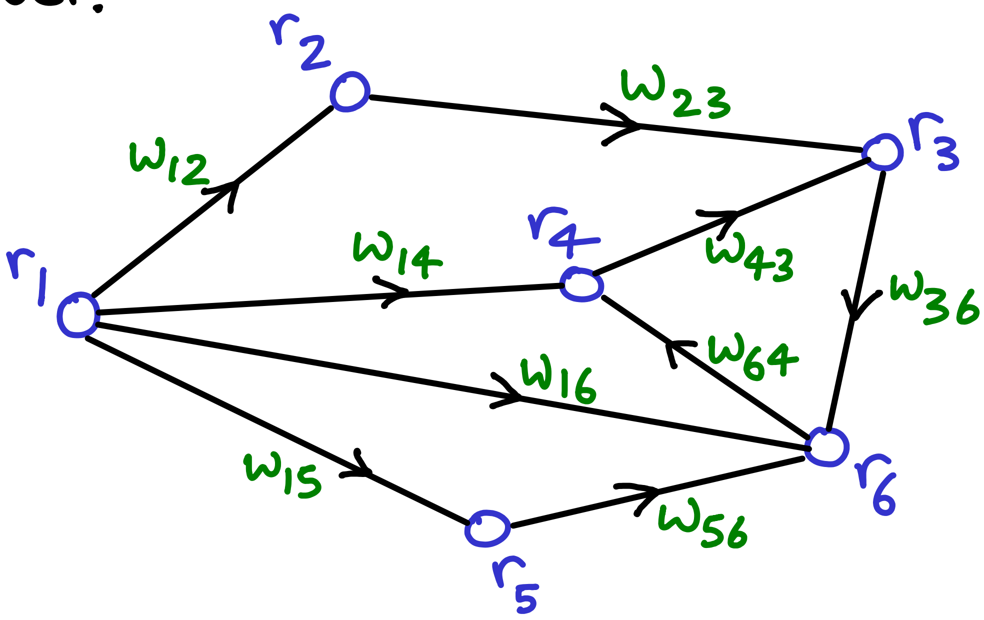
Input weights: w_{ij}

Define $\boxed{w'_{ij} = w_{ij} + r_i - r_j}$



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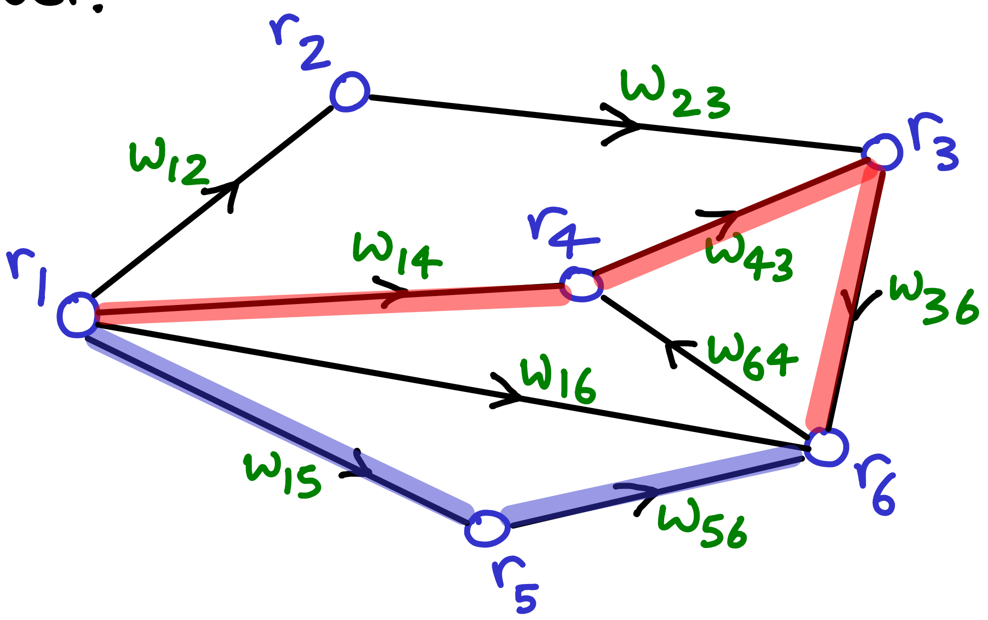
Suppose path $p_1(a \to b)$ is better
than path $p_2(a \to b)$

e.g.: $w_{14} + w_{43} + w_{36} < w_{15} + w_{56}$



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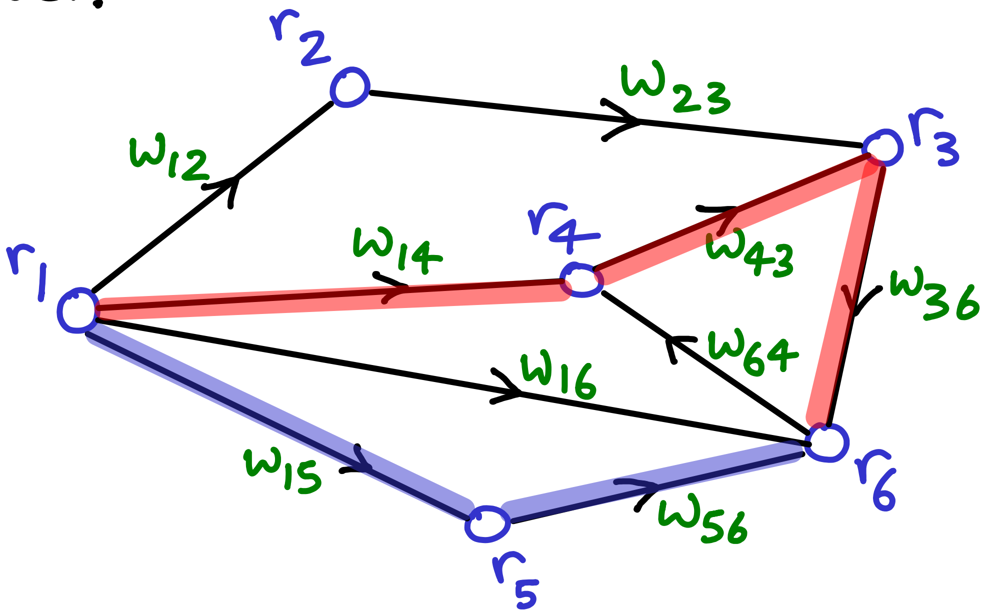

Suppose path $p_1(a \to b)$ is better
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e.g.: $w_{14} + w_{43} + w_{36} < w_{15} + w_{56}$

Compare paths with new weights:

$(w_{14} + r_1 - r_4) + (w_{43} + r_4 - r_3) + (w_{36} + r_3 - r_6)$ vs $(w_{15} + r_1 - r_5) + (w_{56} + r_5 - r_6)$

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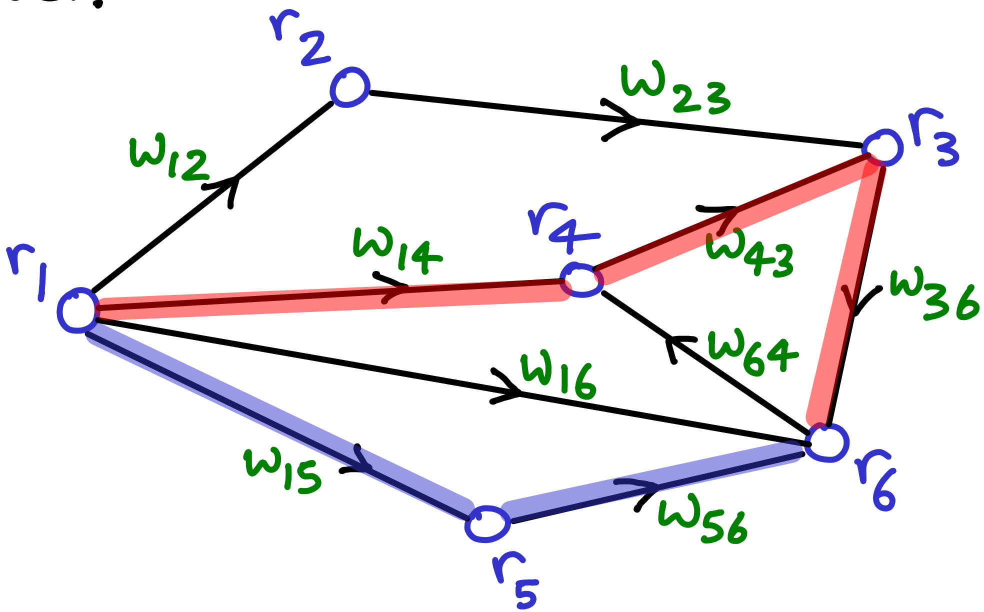

Suppose path $p_1(a \to b)$ is better
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e.g.: $w_{14} + w_{43} + w_{36} < w_{15} + w_{56}$

Compare paths with new weights:

$(w_{14} + r_1 - \cancel{r_4}) + (w_{43} + \cancel{r_4} - r_3) + (w_{36} + r_3 - r_6)$ vs $(w_{15} + r_1 - r_5) + (w_{56} + r_5 - r_6)$

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Input weights: w_{ij}

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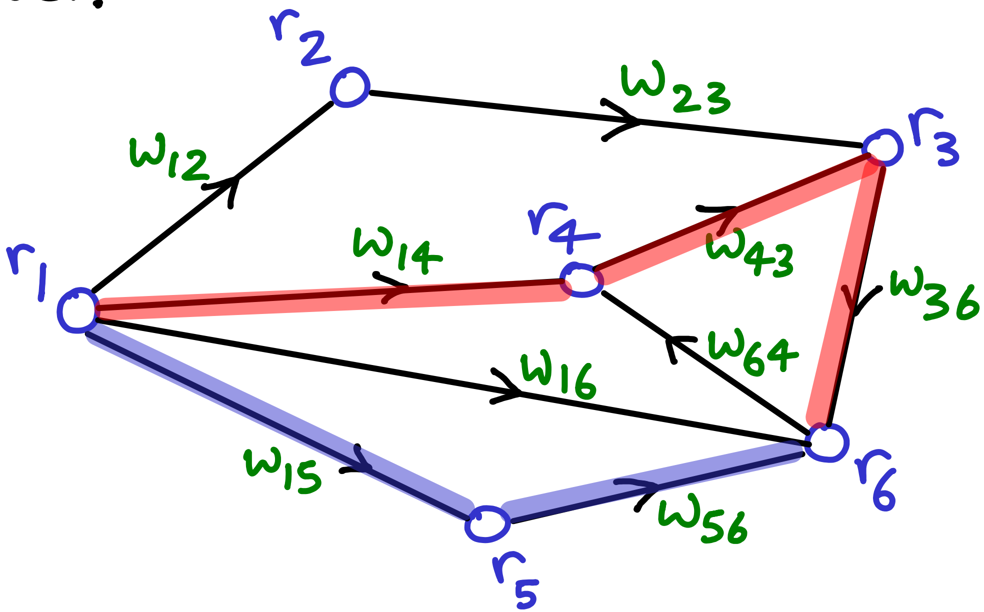

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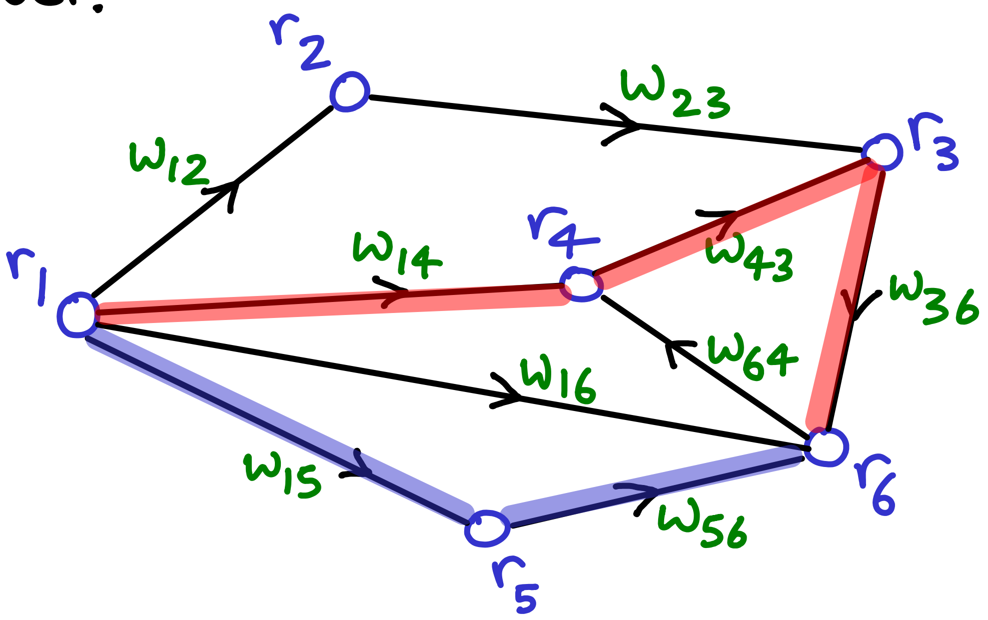
Input weights: w_{ij}

Define $\boxed{w'_{ij} = w_{ij} + r_i - r_j}$



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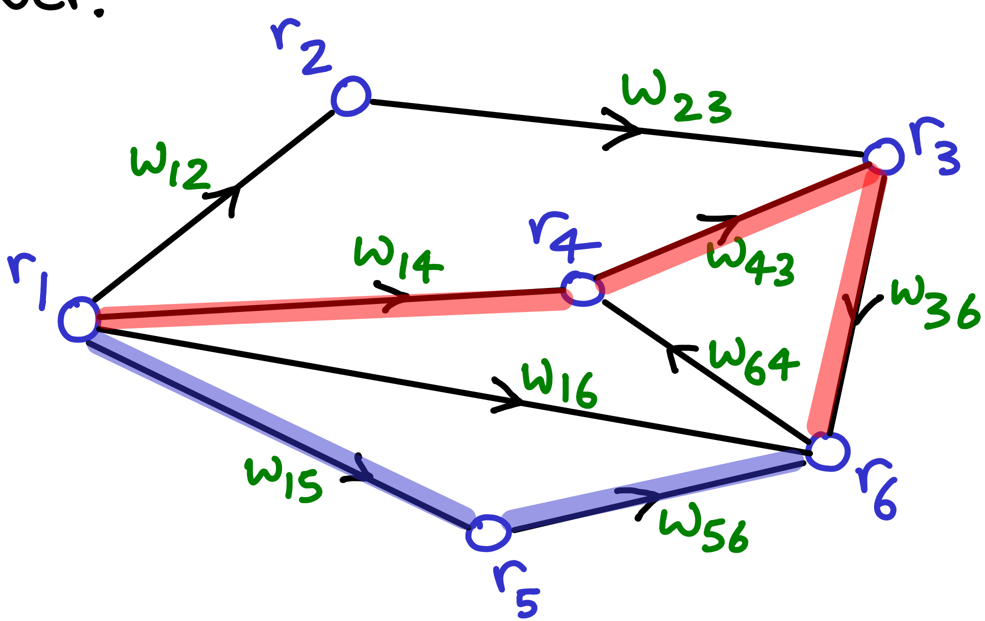
Compare paths with new weights:

$$(w_{14} + \cancel{r_1} - \cancel{r_4}) + (w_{43} + \cancel{r_4} - \cancel{r_3}) + (w_{36} + \cancel{r_3} - \cancel{r_6}) \quad \text{vs} \quad (w_{15} + \cancel{r_1} - \cancel{r_5}) + (w_{56} + \cancel{r_5} - \cancel{r_6})$$

For every vertex k $(1 \leq k \leq V)$
let r_k be some real number.

Input weights: w_{ij}

Define $\boxed{w'_{ij} = w_{ij} + r_i - r_j}$



Suppose path $p_1(a \to b)$ is better
than path $p_2(a \to b)$

e.g.: $w_{14} + w_{43} + w_{36} < w_{15} + w_{56}$

Compare paths with new weights: p_1 still better than p_2

$$(w_{14} + \cancel{r_1} - \cancel{r_4}) + (w_{43} + \cancel{r_4} - \cancel{r_3}) + (w_{36} + \cancel{r_3} - \cancel{r_6}) < (w_{15} + \cancel{r_1} - \cancel{r_5}) + (w_{56} + \cancel{r_5} - \cancel{r_6})$$

For every vertex k ($1 \le k \le V$) let r_k be some real number.

Input weights: w_{ij}

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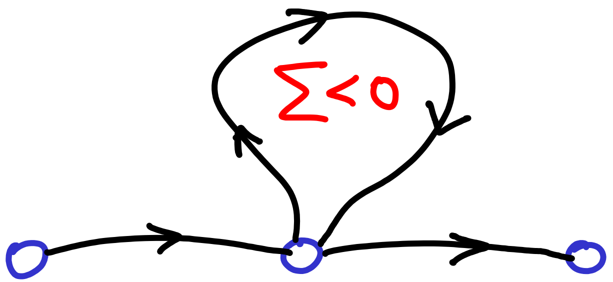


Also, ∃ negative cycle in G′ iff ∃ negative cycle in G

$$\sum_{\text{cycle}} w' = \sum_{\text{cycle}} w + \cancel{\sum_{\text{cycle}} r_i} - \cancel{\sum_{\text{cycle}} r_j}$$

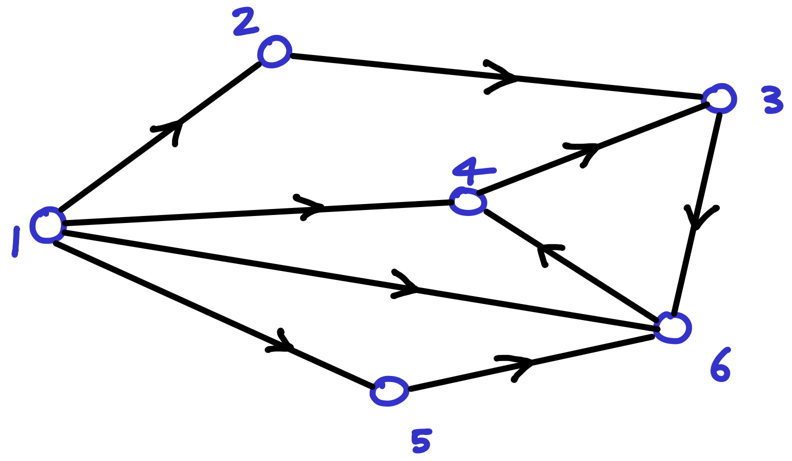
Conclusion: shortest path structure preserved

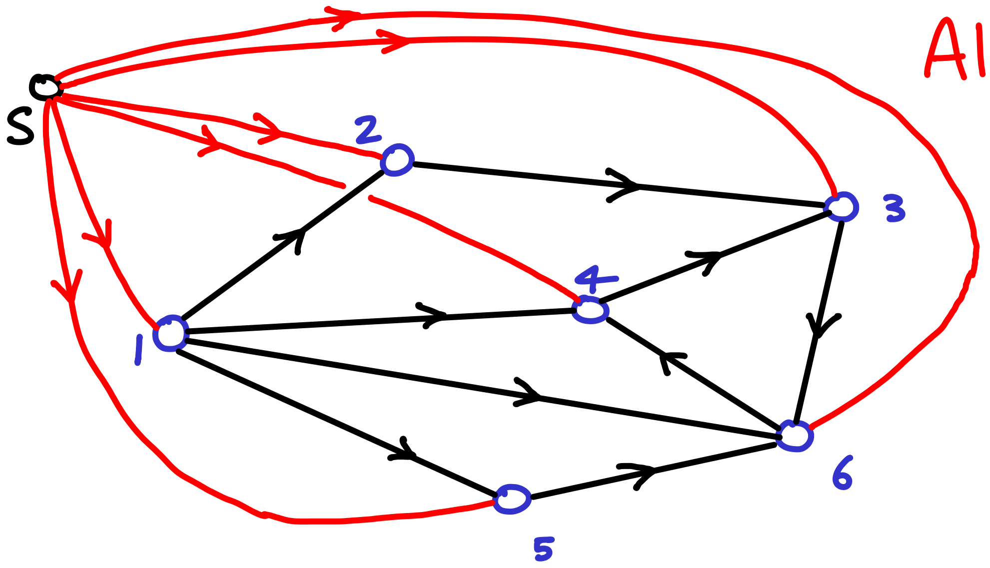
JOHNSON'S ALGORITHM

$\approx$ 1 Bellman-Ford + ADJUST WEIGHTS + $V \cdot$ Dijkstra

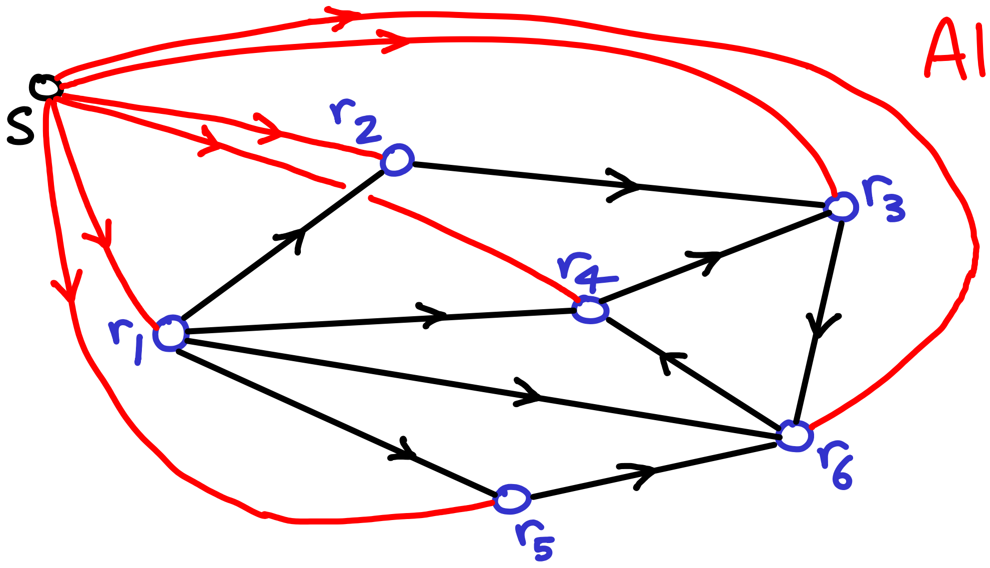
- need weights ≥ 0

✓ need same shortest path structure


2
3
4
1
6
5



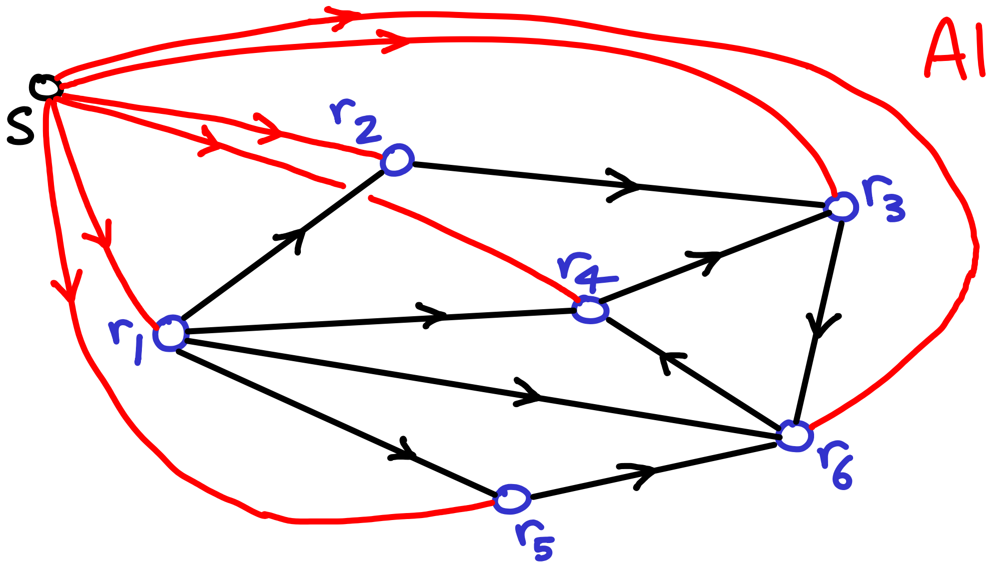
All $w_{sv} = 0$



All $w_{sv} = 0$

Define $r_v = |\text{shortest path } s \to v|$

Use Bellman-Ford



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same shortest path structure

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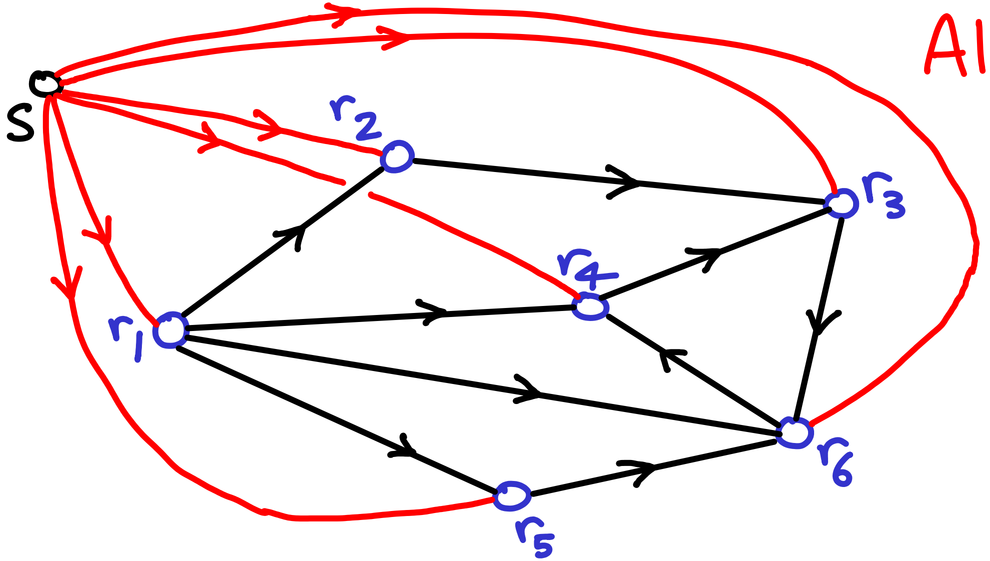


All $w_{sv} = 0$

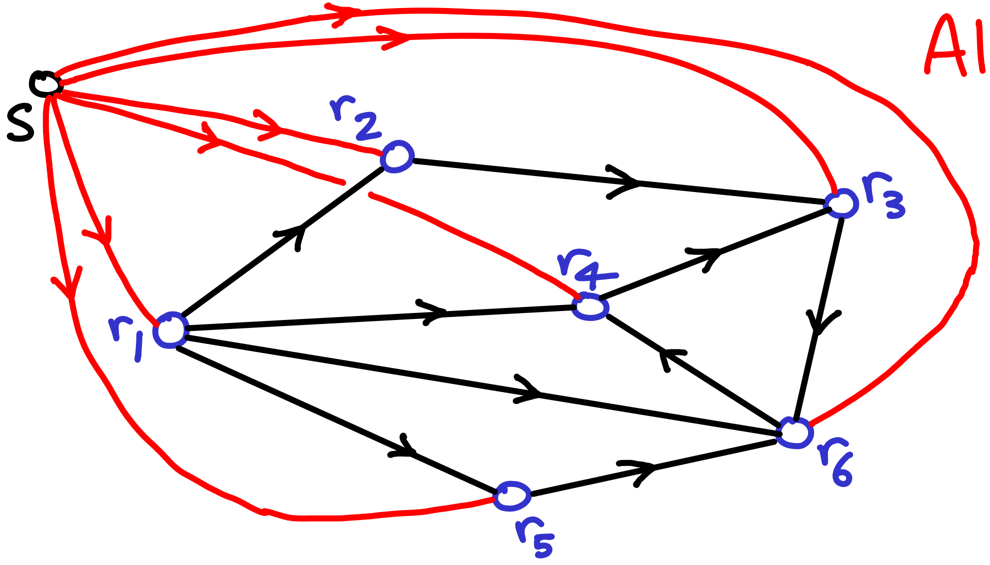
→ same shortest path structure

Define r_v = |shortest path $s \to v$|

Use Bellman-Ford

Assuming no negative cycles detected,

we know B-F ends when no edge-relaxation causes a change.



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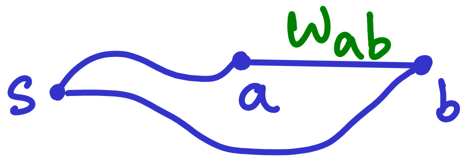
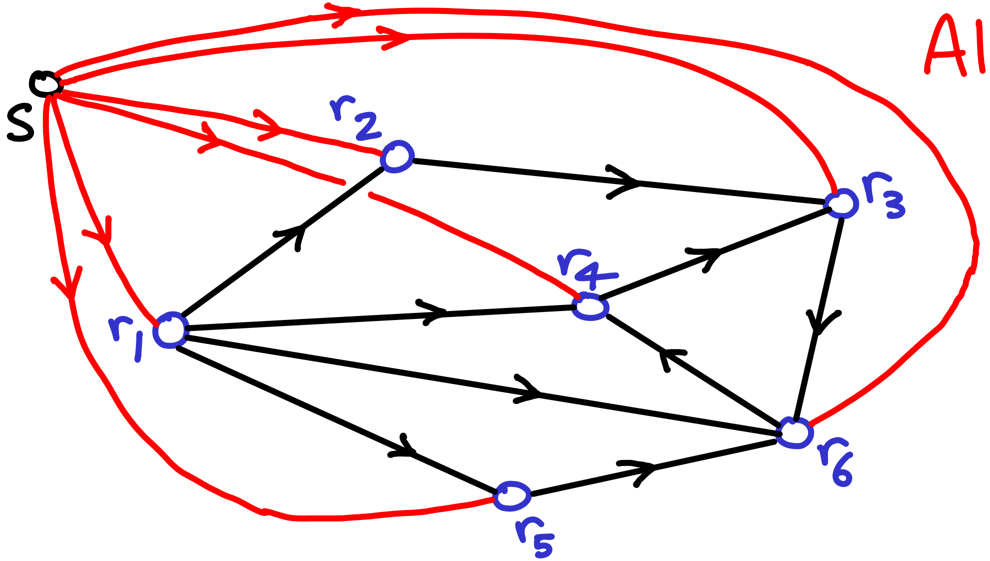


$\hookrightarrow$ for every edge ab, $r_a + w_{ab} \geq r_b$



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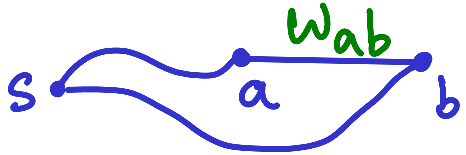
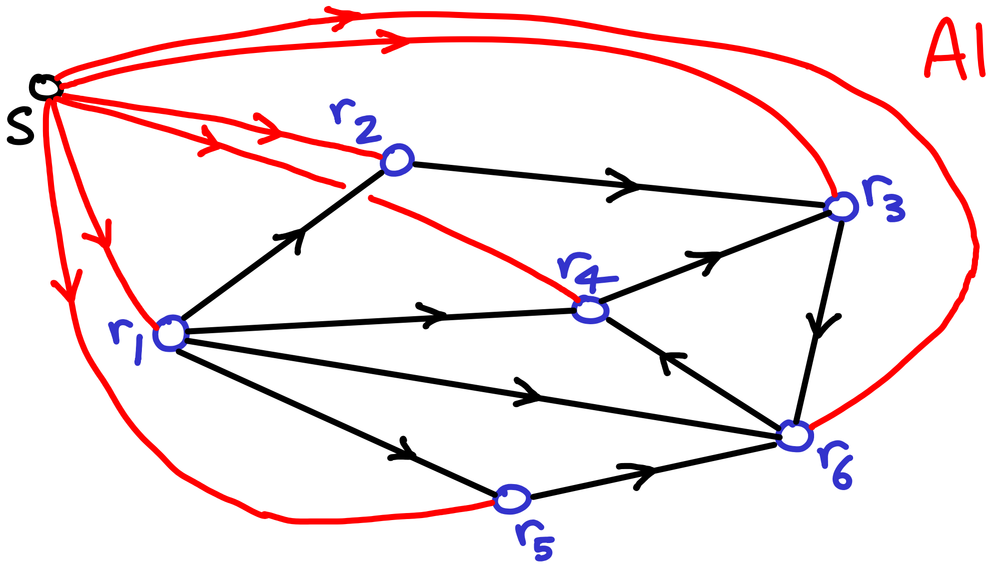


$\hookrightarrow$ for every edge ab, $r_a + w_{ab} \geq r_b$

By definition $w'_{ab} = w_{ab} + r_a - r_b$



All $w_{sv} = 0$

same shortest path structure

Define $r_v = |\text{shortest path } s \to v|$

Use Bellman-Ford

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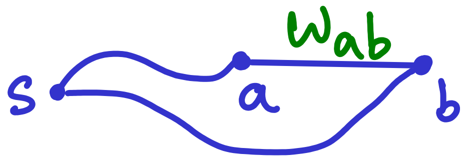
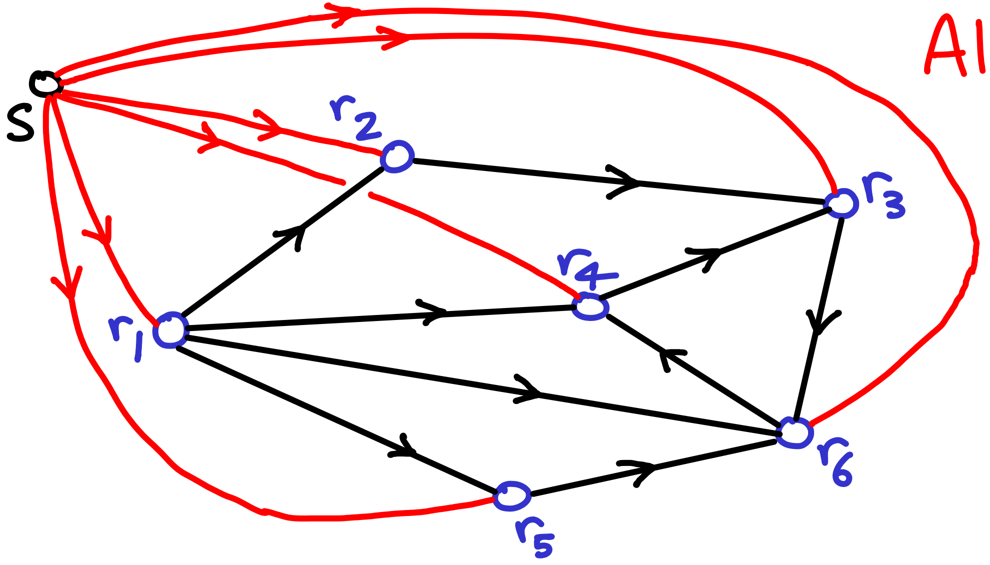


for every edge ab, $r_a + w_{ab} \geq r_b$

$w_{ab} \geq r_b - r_a$

By definition $w'_{ab} = w_{ab} + r_a - r_b$



All $w_{sv} = 0$

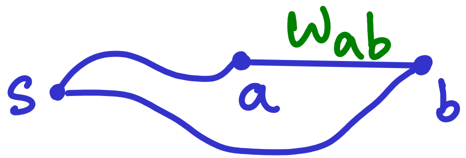
same shortest path structure

Define $r_v = |\text{shortest path } s \to v|$

Use Bellman-Ford

Assuming no negative cycles detected,

we know B-F ends when no edge-relaxation causes a change.



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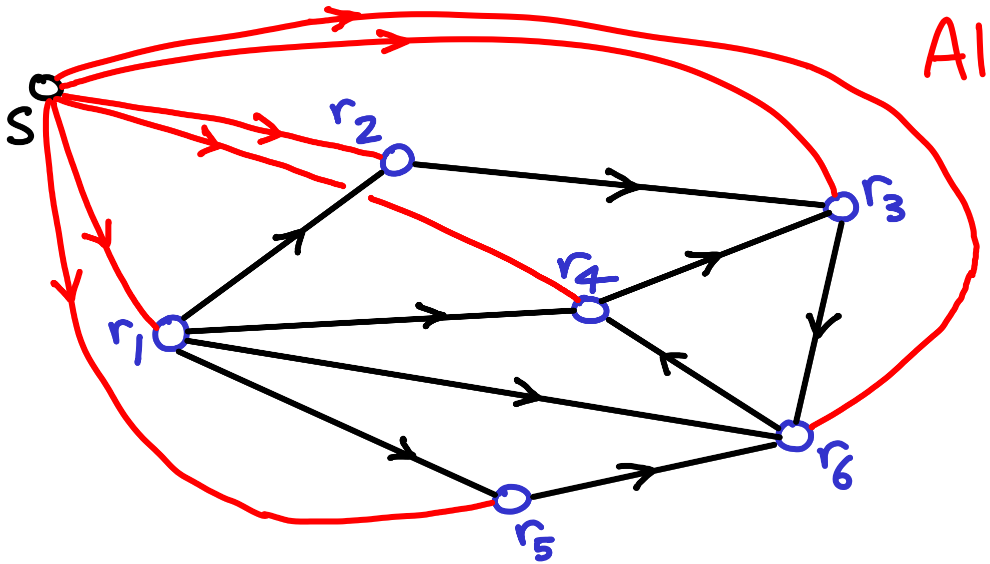

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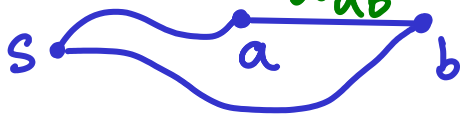

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$w_{ab} \geq r_b - r_a$

By definition $w'_{ab} = w_{ab} + r_a - r_b \geq (r_b - r_a) + r_a - r_b = 0$

JOHNSON'S ALGORITHM

$\approx$ 1 Bellman-Ford + ADJUST WEIGHTS + $V \cdot$ Dijkstra

✓ need weights ≥ 0

✓ need same shortest path structure

JOHNSON'S ALGORITHM

$\approx$ 1 Bellman-Ford + ADJUST WEIGHTS + V · Dijkstra

✓ need weights $\geqslant 0$

✓ need same shortest path structure

Time : $O(V \cdot E)$ + $O(E)$ + V · Dijkstra

JOHNSON'S ALGORITHM

$\approx$ 1 Bellman-Ford + ADJUST WEIGHTS + V · Dijkstra

✓ need weights ≥ 0

✓ need same shortest path structure

$$\text{Time}: \quad O(V \cdot E) \quad + \quad O(E) \quad + \quad \underbrace{V \cdot \text{Dijkstra}}_{V \cdot O(E \log V) \ \text{OR} \ V \cdot O(E + V\log V)}$$

JOHNSON'S ALGORITHM

$\approx$ 1 Bellman-Ford + $\underbrace{\text{ADJUST WEIGHTS}}$ + V · Dijkstra

✓ need weights ≥ 0

✓ need same shortest path structure

Time : $\cancel{O(V \cdot E)} + \cancel{O(E)} + \underbrace{V \cdot \text{Dijkstra}}_{V \cdot O(E \log V) \text{ OR } V \cdot O(E + V \log V)}$

run SSSP, V times	
$V \cdot O(V \cdot E)$	w/ B-F
require weights ≥ 0	
$V \cdot O(E \log V)$	w/ bin. heap Dijkstra (adj. list)
$\cancel{V \cdot \Theta(V^2)}$	w/ "array" Dijkstra (adj. matrix)
$V \cdot O(E + V \log V)$	w/ Fibonacci heap Dijkstra (adj. list)

$\Theta(V^3)$ Floyd-Warshall

allow weights < 0

(still no cycles < 0)

run SSSP, V times

		$\cancel{V \cdot O(V \cdot E)}$	w/ B-F
			require weights ≥ 0
$O(EV\log V)$	Johnson	$\cancel{V \cdot O(E\log V)}$	w/ bin.heap Dijkstra (adj. list)
$\Theta(V^3)$	Floyd-Warshall	$\cancel{V \cdot \Theta(V^2)}$	w/ "array" Dijkstra (adj. matrix)
$O(VE + V^2\log V)$	Johnson	$\cancel{V \cdot O(E + V\log V)}$	w/ Fibonacci heap Dijkstra (adj. list)

allow weights < 0

(still no cycles < 0)