

EULER FORMULA for planar connected graphs: $V - E + F = 2$

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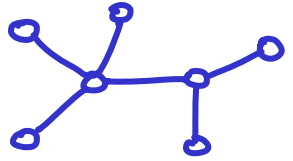
Proof by induction on number of faces:

Base case $\rightarrow F = 1 \rightarrow ?$

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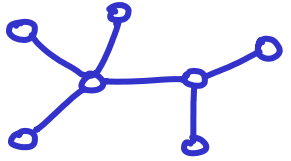
Base case $\rightarrow F = 1 \rightarrow G$ is a tree $\rightarrow ?$



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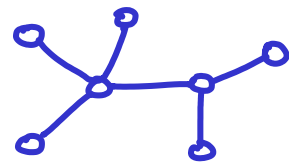
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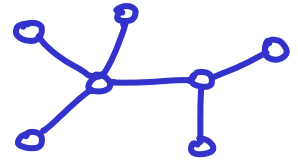


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$$\text{so } (E + 1) - E + 1 = 2 \quad \checkmark$$

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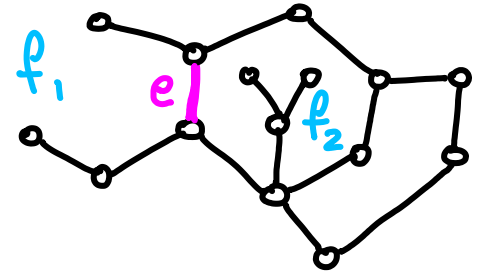
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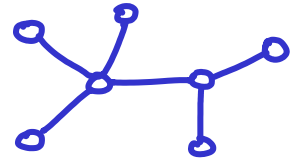
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Given $G=(V,E)$ w/ $F>1$ faces,
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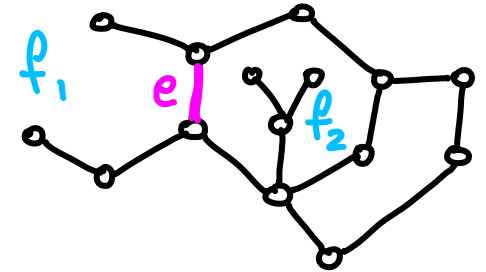
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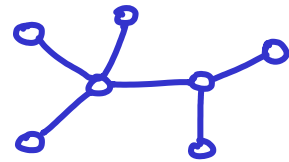
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Given $G=(V,E)$ w/ $F>1$ faces,
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Either f_1 or f_2 is a bounded face



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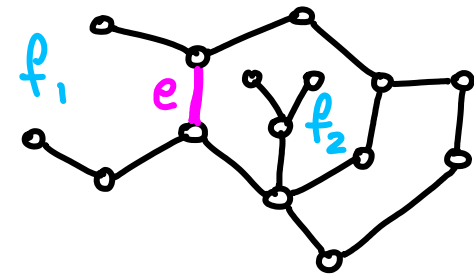
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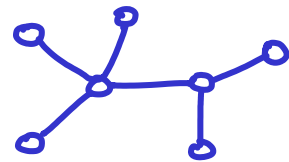
Either f_1 or f_2 is a bounded face,

so e is on a cycle (e is not a cut edge)



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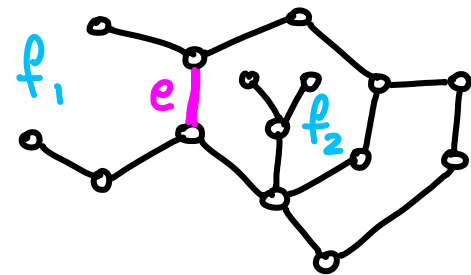
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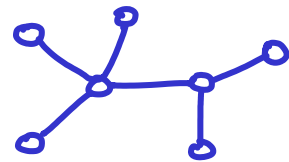
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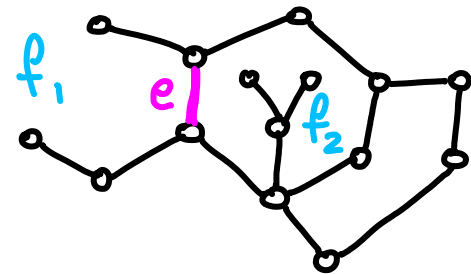
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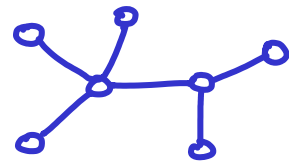
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$$\underbrace{V \quad (E-1) \quad (F-1)}_{G-e}$$

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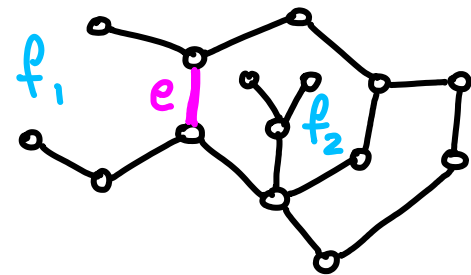
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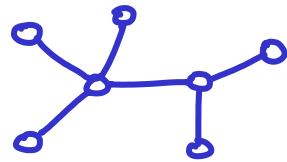
$\hookrightarrow G - e$ is connected & f_1, f_2 merge:

hypothesis

$$V - (E - 1) + (F - 1) = 2$$

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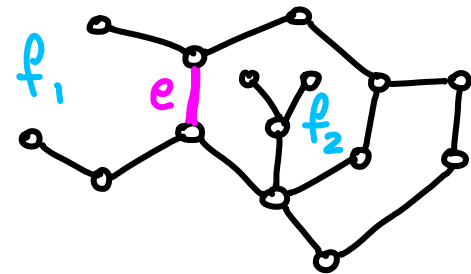
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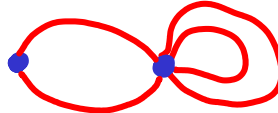


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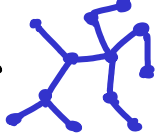
$\hookrightarrow V - E + F = 2 \checkmark$

Note that this also holds for multigraphs 

Euler formula $V - E + F = 2$

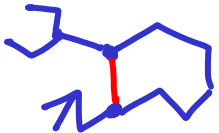
$V - E + F = 2$ applies to any connected planar graph (in fact, to convex polyhedra) by projection

Induction on faces:

$F=1$:  : tree.
 $E = V - 1$

$F > 1$:

Remove an edge between
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Remains connected.

$F \rightarrow F - 1$ $E \rightarrow E - 1$

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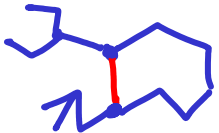
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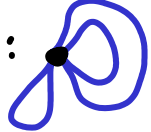
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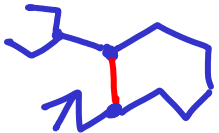
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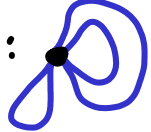
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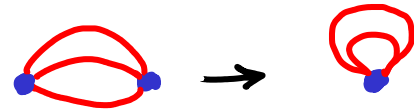
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Contract edge $x \neq y$

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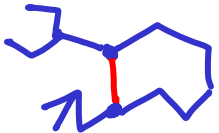
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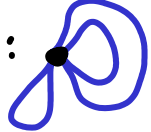
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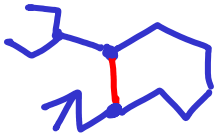
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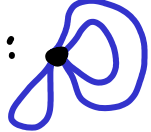
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$F \rightarrow F - 1$ $E \rightarrow E - 1$

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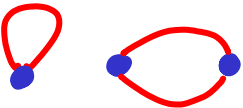
$E=0$:  : one vertex
one face

$E \geq 1$: if $x \neq y$ contract as before

else  remove as before

$E \rightarrow E - 1$ & $F \rightarrow F - 1$
OR
 $V \rightarrow V - 1$

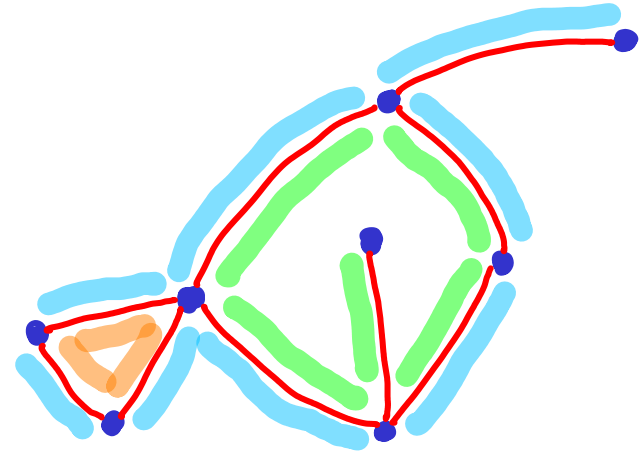
use the Euler formula $V - E + F = 2$
to show that a connected plane graph has $E \leq 3V - 6$ for $V \geq 3$

Not allowed: 

use the Euler formula $V - E + F = 2$

to show that a connected plane graph has $E \leq 3V - 6$

Every edge belongs to 1 or 2 faces $\sum_{\text{all faces}} e \leq 2E$



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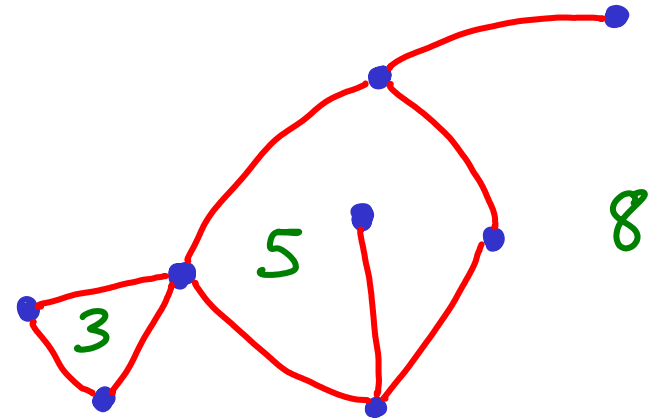
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Every face has ≥ 3 edges (for $V > 3$)

$$\sum_{\text{all faces}} e \geq 3F$$



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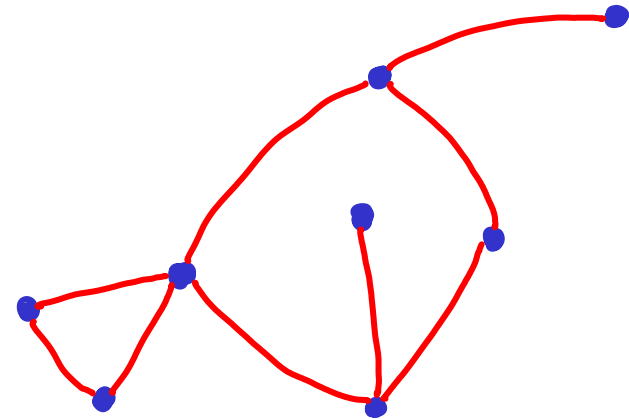
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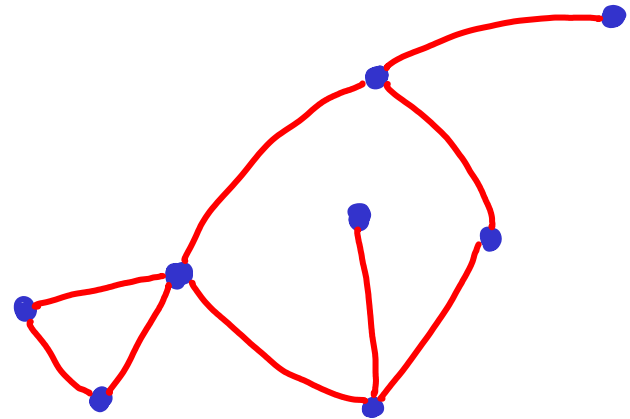
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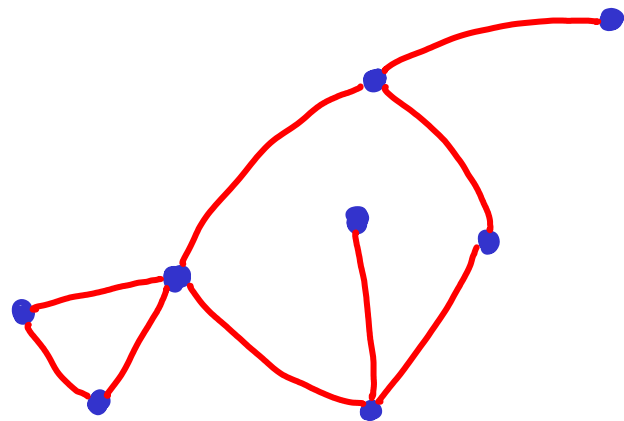
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$$2E \geq 3F$$

$$E - F = V - 2$$

$$E - \frac{2E}{3} \leq V - 2$$

$$E \leq 3V - 6$$



use the Euler formula $V - E + F = 2$

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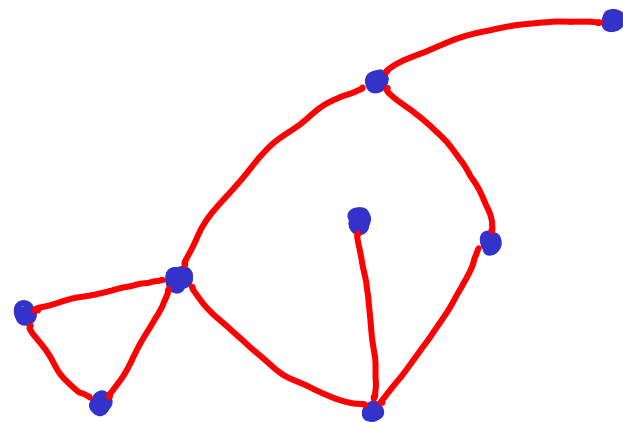
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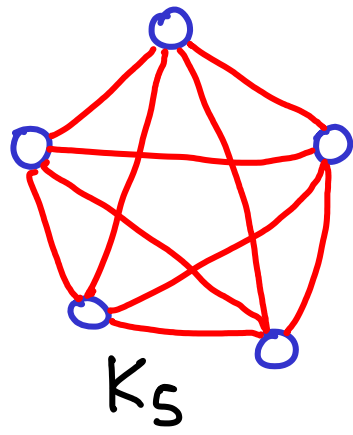
$$\frac{3F}{2} - F \leq V - 2$$

$$F \leq 2V - 4$$



$$E \leq 3V - 6$$

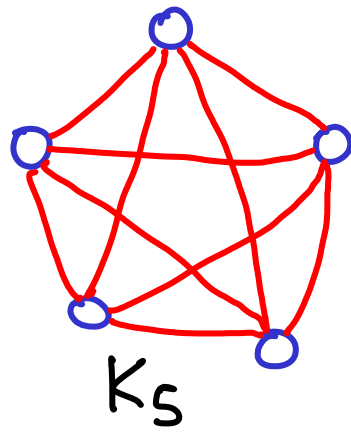
$$E \leq 3V - 6$$



$$E \leq 3V - 6$$

$$10 \leq 15 - 6$$

!!!

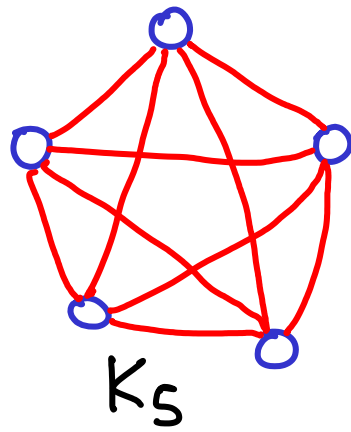


Not planar

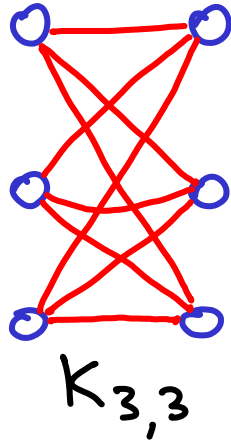
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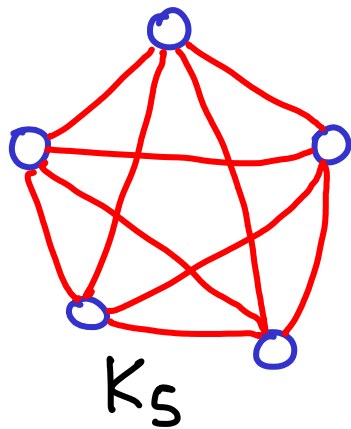
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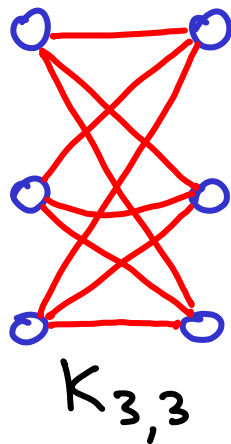
$$10 \leq 15 - 6$$

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$$E \leq 3V - 6$$

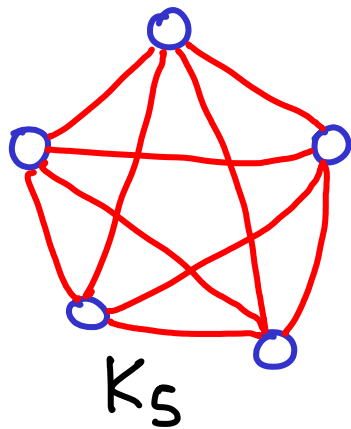
$$9 \leq 18 - 6 \quad \text{ok!}$$



$$E \leq 3V - 6$$

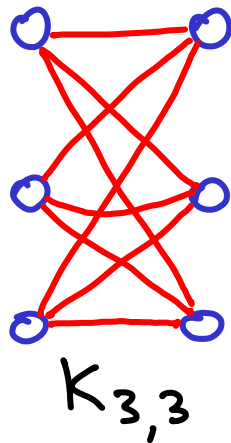
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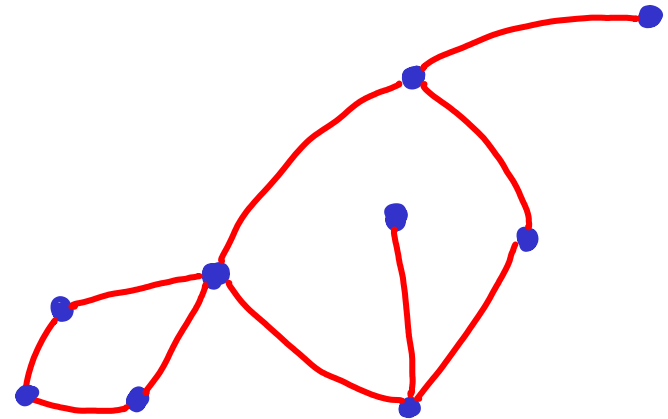


not iff

All planar graphs have $E \leq 3V - 6$
 Some non-planar graphs can too

$$V - E + F = 2$$

What if G has no triangles?



$$V - E + F = 2$$

What if G has no triangles?

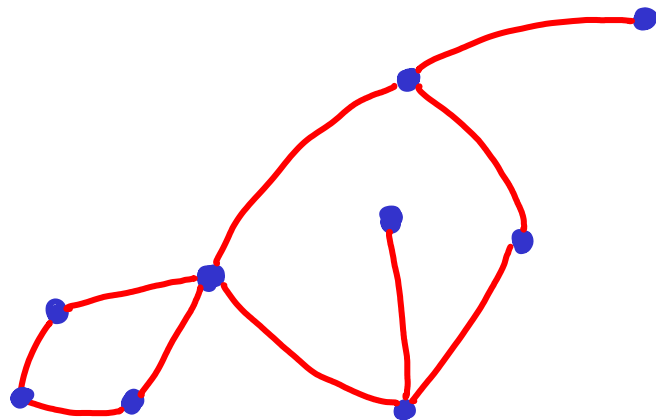
Every edge belongs to 1 or 2 faces

Every face has ~~≥ 3~~ ^{≥ 4} edges (for $V > 4$)

$$\sum_{\text{all faces}} e \leq 2E$$

$$\sum_{\text{all faces}} e \geq \underline{\underline{4F}}$$

$$\underline{\underline{E \geq 2F}}$$



$$V - E + F = 2$$

What if G has no triangles?

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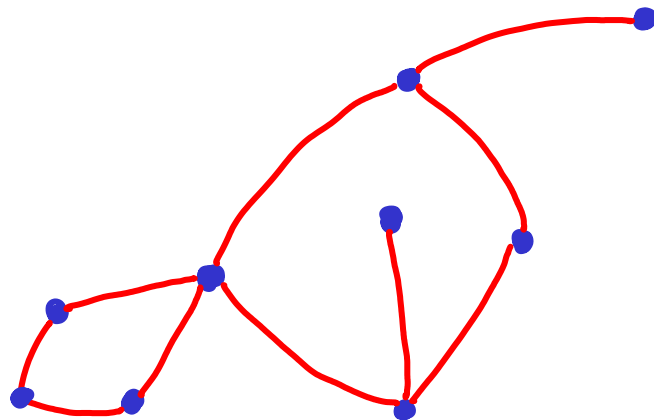
$$\underline{\underline{E \geq 2F}}$$

$$E - F = V - 2$$

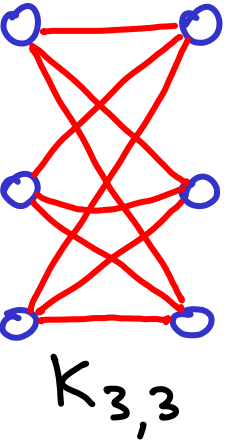
$$E - \frac{E}{2} \leq V - 2$$

$$\underline{\underline{E \leq 2V - 4}}$$

instead of $\leq 3V - 6$



triangle free



for triangle free:

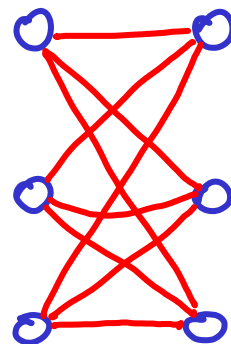
$$E \leq 2V - 4$$

$$9 \leq 2 \cdot 6 - 4$$

!!!

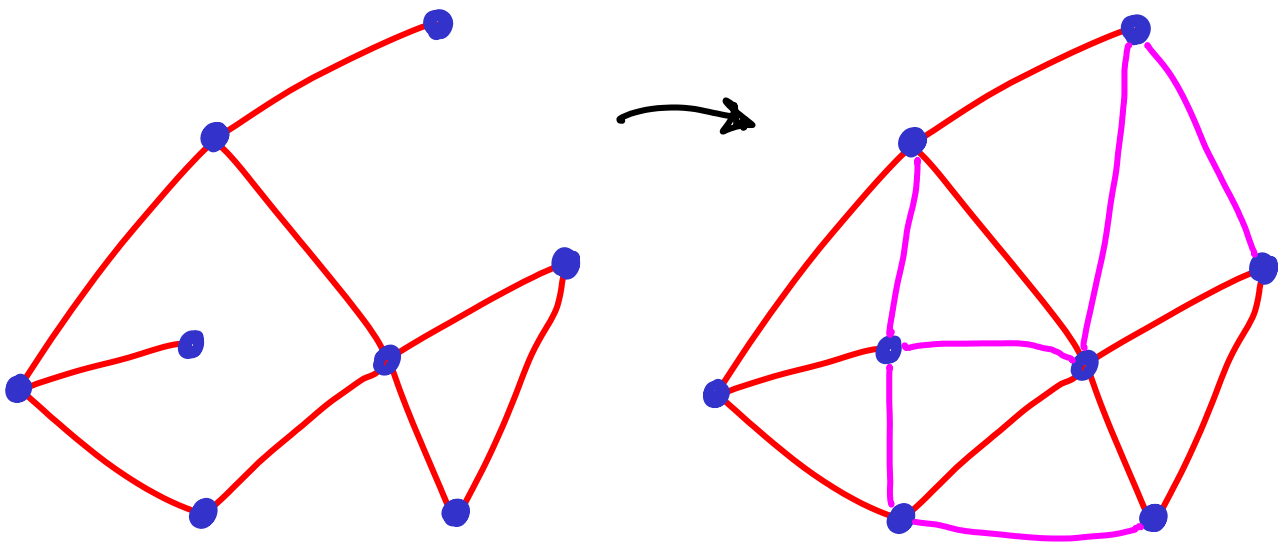
NOT PLANAR

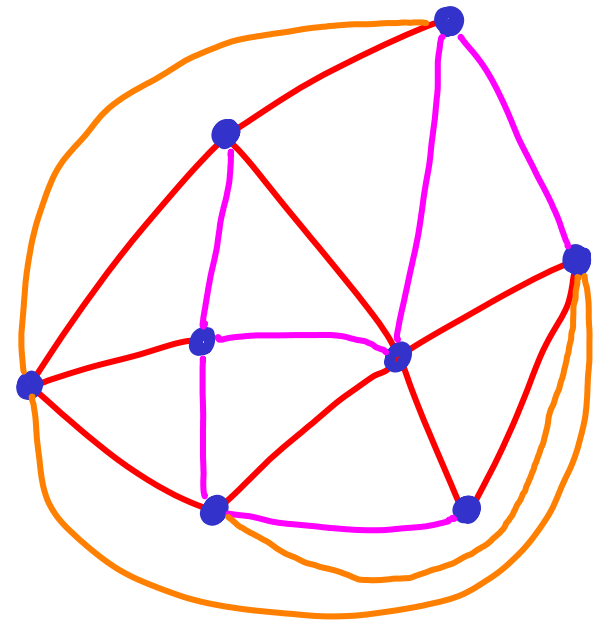
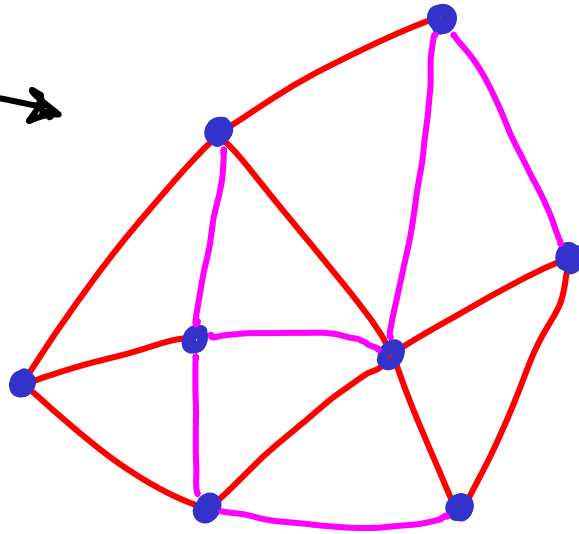
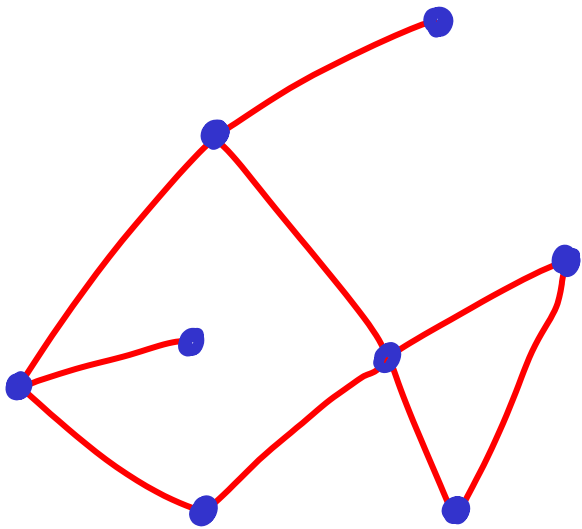
$$V=6, E=9$$



$K_{3,3}$

TRIANGULATIONS

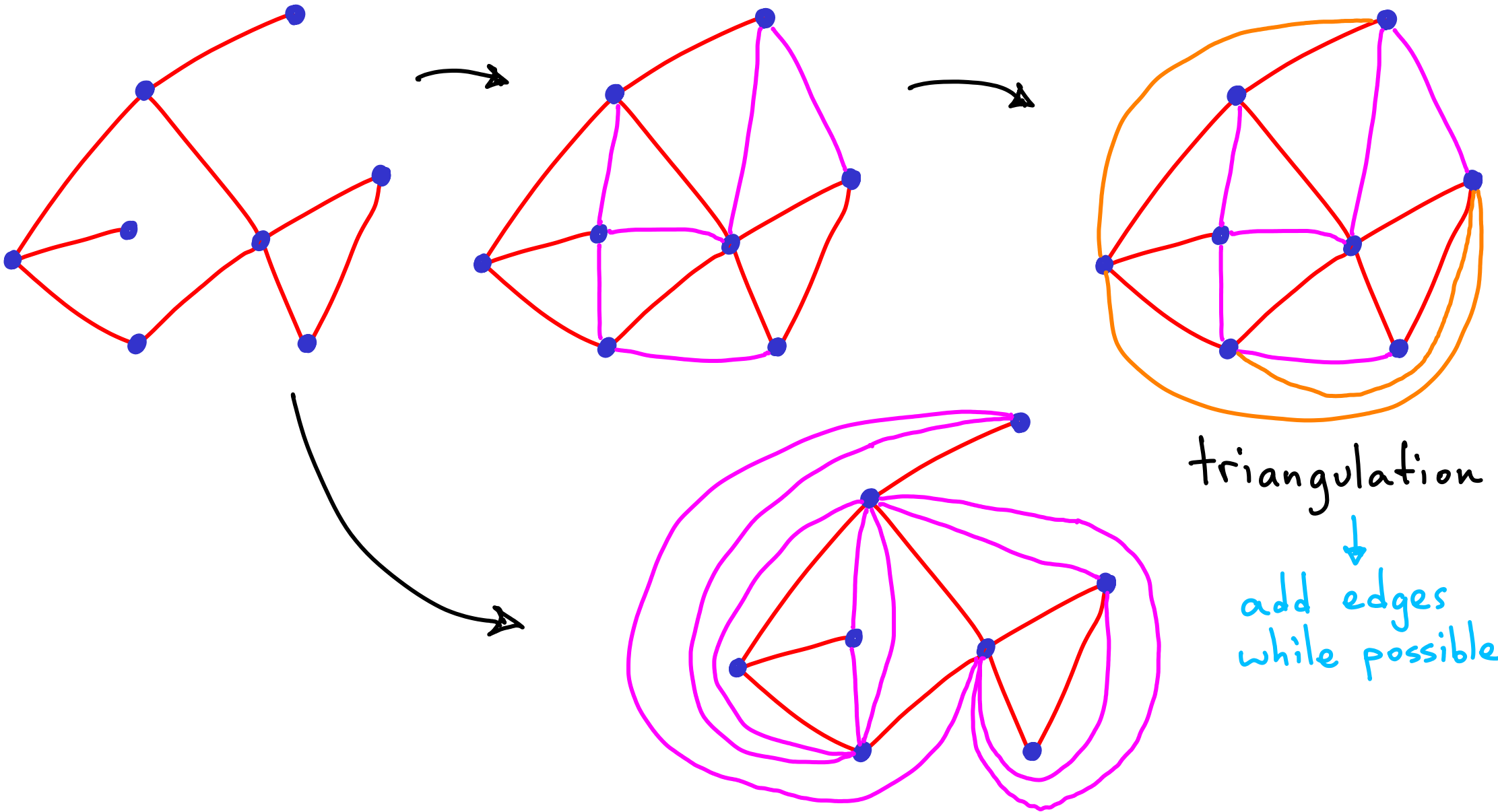




triangulation



add edges
while possible

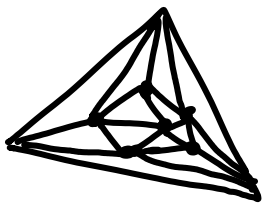


triangulation

↓
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while possible

$$\underline{\underline{E=3V-6}}$$

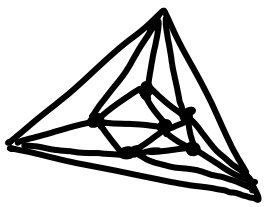
Why?



{ Assume outer face is a triangle

$$\underline{\underline{E = 3V - 6}}$$

Why?



{ Assume outer face is a triangle

Every edge belongs to 1 or 2 faces

$$\sum_{\text{all faces}} e \leq 2E$$

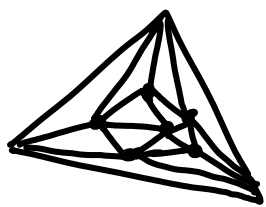
Every face has ≥ 3 edges (for $V > 3$)

$$\sum_{\text{all faces}} e \geq 3F$$

$$2E \geq 3F$$

$$\underline{\underline{E=3V-6}}$$

Why?



Assume outer face is a triangle

Every edge belongs to ~~1~~ or 2 faces

$$\sum_{\text{all faces}} e \stackrel{=}{\neq} 2E$$

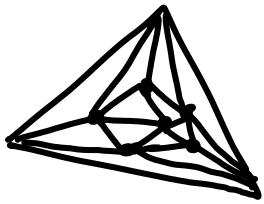
Every face has ~~≠~~⁼ 3 edges (for $V > 3$)

$$\sum_{\text{all faces}} e \stackrel{=}{\neq} 3F$$

$$2E \stackrel{=}{\neq} 3F$$

$$\underline{\underline{E = 3V - 6}}$$

Why?



{ Assume outer face is a triangle

$$\underline{V - E + F = 2}$$

Every edge belongs to ~~1~~ or 2 faces

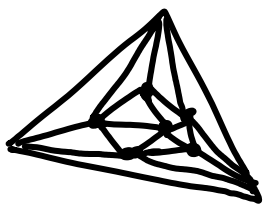
$$\left. \begin{array}{l} \sum_{\text{all faces}} e \stackrel{=}{\neq} 2E \\ \sum_{\text{all faces}} e \stackrel{=}{\neq} 3F \end{array} \right\} 2E \stackrel{=}{\neq} 3F$$

Every face has ~~≠~~ 3 edges (for $V > 3$)

$$E - F = V - 2$$

$$\underline{\underline{E = 3V - 6}}$$

Why?



{ Assume outer face is a triangle

$$\boxed{V - E + F = 2}$$

Every edge belongs to ~~1~~ or 2 faces

$$\sum_{\text{all faces}} e \stackrel{=}{\neq} 2E$$

Every face has ~~3~~ edges (for $V > 3$)

$$\sum_{\text{all faces}} e \stackrel{=}{\neq} 3F$$

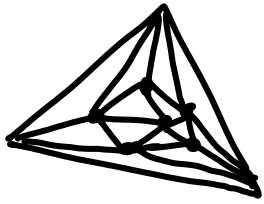
$$2E \stackrel{=}{\neq} 3F$$

$$E - F = V - 2$$

$$E - \frac{2E}{3} = V - 2$$

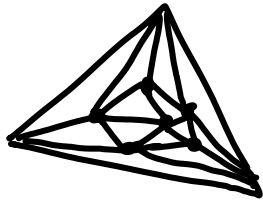
$$E = 3V - 6$$

$$\underline{\underline{E=3V-6}}$$



What is the average degree of a triangulation?

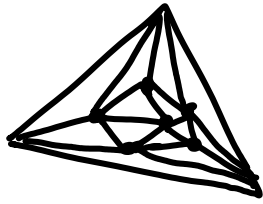
$$\underline{\underline{E=3V-6}}$$



What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^V d(v_i)$$

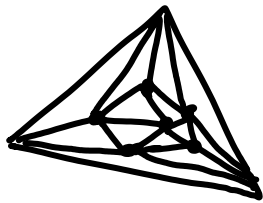
$$\underline{\underline{E = 3V - 6}}$$



What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^V d(v_i) = \frac{1}{V} \cdot 2E$$

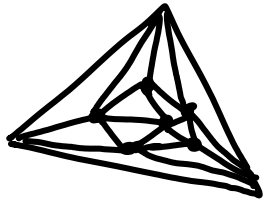
$$\underline{\underline{E = 3V - 6}}$$



What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^V d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} < \underline{\underline{6}}$$

$$\underline{\underline{E = 3V - 6}}$$

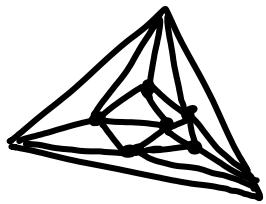


What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^V d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} < \underline{\underline{6}}$$

↳ Every triangulation has a vertex w/ degree ≤ 5

$$\underline{\underline{E = 3V - 6}}$$



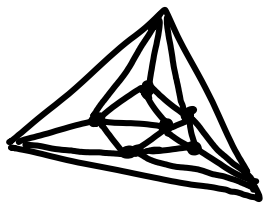
What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^V d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} < \underline{\underline{6}}$$

↳ Every triangulation has a vertex w/ degree ≤ 5

↳ Immediately applies to any planar graph
(fewer edges)

$$\underline{\underline{E = 3V - 6}}$$

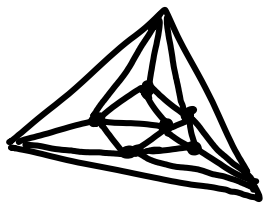


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↳ Every planar graph has a vertex w/ degree ≤ 5

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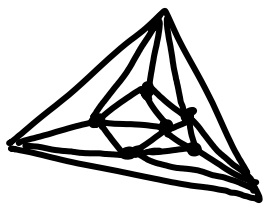
What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^V d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} < \underline{\underline{6}}$$

↳ Every planar graph has a vertex w/ degree ≤ 5

Can we find many low-degree vertices?

$$\underline{\underline{E = 3V - 6}}$$



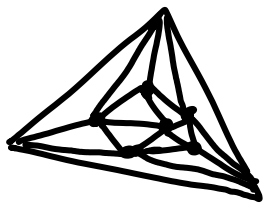
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↳ Every planar graph has a vertex w/ degree ≤ 5

Can we find many low-degree vertices? $\rightarrow$ not if "low" = 5.
what if "low" = 8?

$$\underline{\underline{E = 3V - 6}}$$



What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^V d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} < \underline{\underline{6}}$$

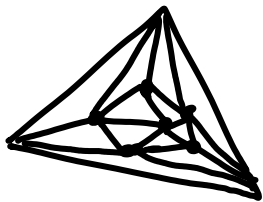
↳ Every planar graph has a vertex w/ degree ≤ 5

Can we find many low-degree vertices? $\rightarrow$ not if "low" = 5.
what if "low" = 8?

$\searrow \frac{V}{2}$ degree-8 $\nearrow$

Prove by contradiction

$$\underline{\underline{E = 3V - 6}}$$



What is the average degree of a triangulation?

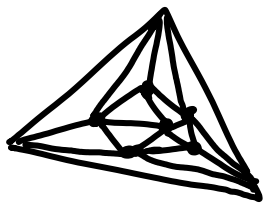
$$\frac{1}{V} \cdot \sum_{i=1}^V d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V-12}{V} < \underline{\underline{6}}$$

↳ Every planar graph has a vertex w/ degree ≤ 5

Can we find many low-degree vertices? $\rightarrow$ not if "low" = 5.
what if "low" = 8?

Say you had $\geq \frac{V}{2}$ vertices w/ degree ≥ 9

$$\underline{\underline{E = 3V - 6}}$$



What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^V d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V - 12}{V} < \underline{\underline{6}}$$

↳ Every planar graph has a vertex w/ degree ≤ 5

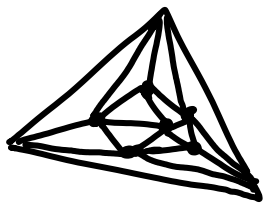
Can we find many low-degree vertices? → not if "low" = 5.
what if "low" = 8?

Say you had $\geq \frac{V}{2}$ vertices w/ degree ≥ 9

↳ $\sum_{d \geq 9} d(v_i) \geq 9 \cdot \frac{V}{2}$

sum degrees of $\frac{V}{2}$ of them

$$\underline{\underline{E = 3V - 6}}$$



What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^V d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V - 12}{V} < \underline{\underline{6}}$$

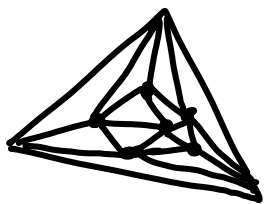
↳ Every planar graph has a vertex w/ degree ≤ 5

Can we find many low-degree vertices? → not if "low" = 5.
what if "low" = 3?

Say you had $\geq \frac{V}{2}$ vertices w/ degree ≥ 9

↳ $\sum_{d \geq 9} d(v_i) \geq 9 \cdot \frac{V}{2}$ } all other vertices have degree ≥ 3
 sum degrees of $\frac{V}{2}$ of them } ↳ $\sum d(v_i) \geq 3 \cdot \frac{V}{2}$

$$\underline{\underline{E = 3V - 6}}$$



What is the average degree of a triangulation?

$$\frac{1}{V} \cdot \sum_{i=1}^V d(v_i) = \frac{1}{V} \cdot 2E = \frac{6V - 12}{V} < \underline{\underline{6}}$$

↳ Every planar graph has a vertex w/ degree ≤ 5

Can we find many low-degree vertices? → not if "low" = 5.
what if "low" = 8?

Say you had $\geq \frac{V}{2}$ vertices w/ degree ≥ 9

↳ $\sum_{d \geq 9} d(v_i) \geq 9 \cdot \frac{V}{2}$
sum degrees of $\frac{V}{2}$ of them

} all other vertices have degree ≥ 3
↳ $\sum d(v_i) \geq 3 \cdot \frac{V}{2}$

} $\sum + \sum \geq 6V$
contradiction
↓
always have
 $\geq \frac{V}{2}$ w/ deg ≤ 8

