Introduction to Matroids

Guest lecture in COMP150-Graph Theory
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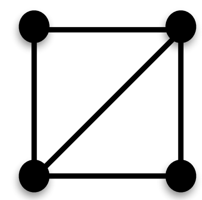
What is a matroid?

- A mathematical structure that generalizes concepts from graph theory, linear algebra, etc.
- Introduced in the 1930s by Whitney, Nakasawa, MacLane, and van der Warden
- Based on the concept of *hereditary system*
- This lecture assumes matroids are finite to avoid problems with duality, though recent work by Bruhn, Diestel, Kriesell, Pendavingh, and Wollan (2013), has extended the theory to infinite objects called *B-matroids*

So what is a hereditary system?

- A hereditary system, M, on a set, E (the *ground* set), consists of a nonempty collection, I_M, of subsets of E with the property that every member of I_M also has all its subsets in I_M.
- Hereditary systems are also called *independence systems* or *abstract simplicial complexes*
- The members of I_M are called *independent sets*.
- There may be several ways of specifying I_M. These are called *aspects* of M.

An example of a hereditary system and independent sets

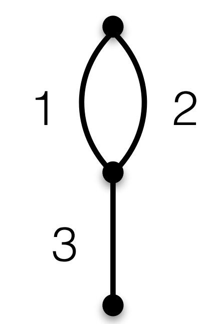
- E = edges of the kite 
- I_M = sets of edges with no cycle
- every set with more than three edges is dependent
- two more dependent sets = ??
- maximal independent sets = spanning trees

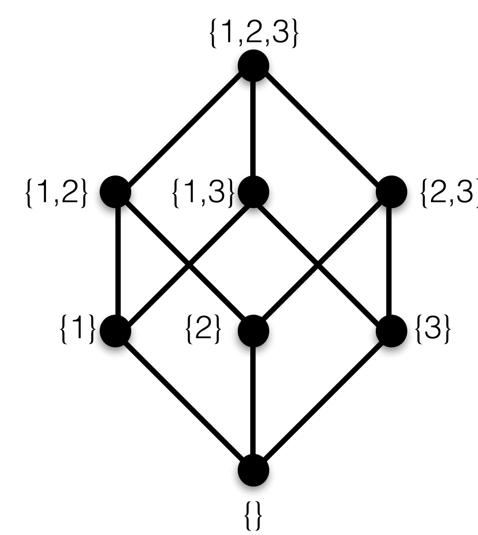
Some more terminology for independent sets

- B_M = *bases* = maximal independent sets
- C_M = *circuits* = minimal dependent sets
- r_M (X a subset of E) = *rank* = maximum size of an independent set
  $= \max\{ |Y| : Y \subseteq X, Y \in I_M \}$
- $r_M(\ )$ satisfies the following two properties (Lemma 8.2.17):
  (r1) The rank of the empty set is zero
  (r2) If $X \subseteq E$ and $e \in E$, then $r_M(X) \le r_M(X + e) \le r_M(X) + 1$

Another example of a hereditary system

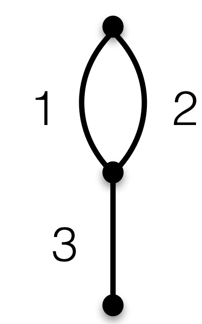
- I_M = sets of edges with no cycle
- dependent sets = ??
- C_M = circuits = ??
- B_M = bases = ??
- $r_M(\,)$ = rank = ??

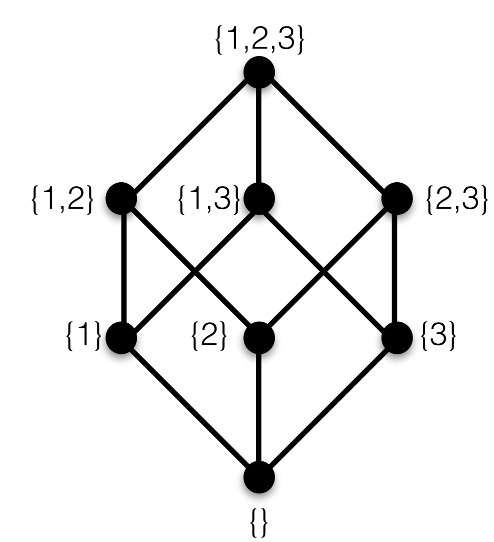




Another example of a hereditary system

- I_M = sets of edges with no cycle
- dependent sets = {1,2} and {1,2,3}
- C_M = circuits = {1,2}
- B_M = bases = {1,3} and {2,3}
- $r_M(\ )$ = rank = size (if independent)
  r({1,2}) = 1 r({1,2,3}) = 2





A matroid is a hereditary system with an additional property

- One such property is the *base exchange property* :
- if B_1 and B_2 are bases, then for every e in $B_1 - B_2$
  there is an f in $B_2 - B_1$ so that $B_1 - \{e\} + \{f\}$ is a base
- For example, in a connected graph the bases are spanning trees
  Deleting an edge from a spanning tree disconnects it
  The two components can be reconnected using a different edge from another spanning tree
- One consequence is that all bases have the same size, which you already know to be true of spanning trees (or spanning forests in the case of graphs with more than one component)

A matroid is a hereditary system with an additional property

- Another such property is the *(weak) absorption property* :
- if X is a subset of E and f and g are members of E with
  r(X) = r(X + e) = r(X + f), then
  r(X) = r(X + e + f)
-
-

A matroid is a hereditary system with an additional property

- Another such property is the *(weak) absorption property* :
- if X is a subset of E and *f* and *g* are members of E with
  $r(X) = r(X + e) = r(X + f)$, then
  $r(X) = r(X + e + f)$
- There must be a *strong absorption property* :
- if X and Y are subsets of E with
  $r(X) = r(X + e)$ for all *e* in Y, then
  $r(X \cup Y) = r(X)$

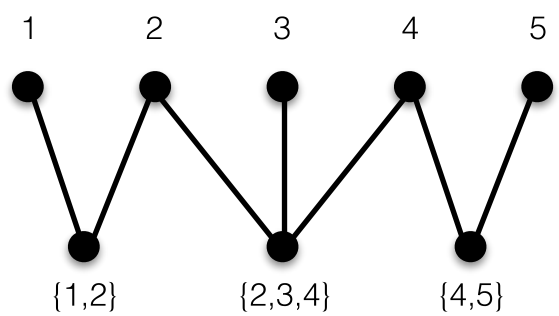
A matroid is a hereditary system with an additional property

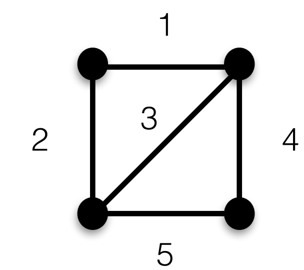
- A fourth such property is the *augmentation property* :
- if I_1 and I_2 are independent sets with $|I_1| > |I_2|$, then
  $I_2 + e$ is independent for some e in $I_1 - I_2$
-
-

A matroid is a hereditary system with an additional property

- A fourth such property is the *augmentation property* :
- if I_1 and I_2 are independent sets with $|I_1| > |I_2|$, then
  $I_2 + e$ is independent for some e in $I_1 - I_2$
- Theorem 3.1.10 (Berge, 1957): A matching M in a graph G is a maximum matching in $G \Leftrightarrow G$ has no M-augmenting path
- so if G has an M-augmenting path then there is a matching M' with $|M'| > |M|$ and $M' \triangle M$ contains an M-augmenting path

Transversal matroids from matchings in a bipartite graph





- For a bipartite graph, G = (U, V, E) let the elements of a matroid be the vertices in U and the independent sets be sets of endpoints of matchings

- This matroid satisfies the augmentation property and is called a *transversal matroid*

- In the example above, the matroid is isomorphic to the kite with independent sets being acyclic subsets of edges

Graphic matroids

- The *cycle matroid* of a graph, G, is the matroid with ground set E(G) and circuits (minimal dependent sets) given by the cycles of G

- A matroid that can be defined in this way is called a *graphic matroid*

- Not every matroid is graphic

Vectorial matroids

- The ground set, E, is a set of vectors, $\{x_i\}$, in a vector space
- I = subsets of E that are linearly independent
- Dependent sets must have $\sum c_i x_i = 0$ with some c_i being nonzero
- Circuits are sets of x_i with $\sum c_i x_i = 0$ forcing all $c_i \neq 0$
- Not every matroid is vectorial
- The column matroid of this matrix is the cycle matroid of the kite:

0	0	0	1	1
0	1	1	0	1
1	1	0	0	0

Uniform matroids and free matroids

- $U_{k,n}$ is the *uniform matroid* of rank k defined on any ground set of size n with bases being all of the subsets of size k
- so the independent sets are all of the subsets of size at most k
- If $n = k$ this is called the *free matroid* of rank n
- Uniform matroids may or may not be graphic and graphic matroids may or may not be uniform (Exercise 8.2.6 in West)

Partition matroids

- If E is partitioned into distinct blocks E_1, E_2, … E_k, the *partition matroid* induced by this partition is the matroid with independent sets having at most one element in each block of the partition

- In a directed graph, each edge has a head and a tail, so the edges can be partitioned in two ways, called the *head partition* and the *tail partition*

Matroids and greedy algorithms

- Matroids can be defined by the *greedy algorithm* property:
- For any nonnegative weight function on the ground set, the greedy algorithm selects an independent set of maximum total weight
- greedy(matroid M) returns: independent set I

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I ← empty;  E ← M.E;
while (E is nonempty) {
    e ← an element of E of maximum weight;
    remove e from E;
    if (I + e is independent) then I ← I + e;
}
return I
```

Matroids and matchings

- Given a bipartitite graph G = (U, V, E) it would seem natural to define a matroid by defining the independent sets to be the matchings of G
- This doesn't always work (Exercises 8.2.1 and 8.2.2 in West)
- Suppose G is a directed graph with all edges directed from U to V, then any matching is contained in both the head partition and the tail partition
- This motivates the definition of the intersection of two matroids (which may not be a matroid)

Matroid intersection

- Given matroids M_1 and M_2, with independent sets I_1 and I_2, $M_1 \cap M_2$ is the hereditary system with independent sets being those sets that are independent in both M_1 and M_2

- Although $M_1 \cap M_2$ is not a matroid in general, it does have the following property, proved by the Matroid Intersection Theorem:
  $\max\{|I| : I \in I_1 \cap I_2\} = \min \{r_1(A) + r_2(\bar{A})\}$
  where the min is over all subsets of the ground set and $\bar{A}$ is the complement of A with respect to the ground set

- The matroid intersection algorithm (Papdimitriou and Steiglitz) solves this max-min problem in time $O(|E|^3 C(|E|))$ where $C(|E|)$ is the time required for matroid queries

Problems with intersections of more than two matroids

- Intersecting more than two matroids can lead to NP-complete problems, so there's not much hope for a polynomial-time algorithm for solving matroid intersection problems of higher order

- Papadimitriou and Steiglitz show how to define the Hamiltonian path problem as the intersection of three matroids

The (weak) elimination property

- The *weak elimination property* :
- If C_1 and C_2 are distinct circuits and $x \in C_1 \cap C_2$ then there is another circuit contained in $C_1 \cup C_2 - x$
-
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The (weak) elimination property

- The *weak elimination property* :
- If C_1 and C_2 are distinct circuits and $x \in C_1 \cap C_2$ then there is another circuit contained in $C_1 \cup C_2 - x$
- There must be a *strong elimination property* :
- If C_1 and C_2 are circuits with $x \in C_1 \cap C_2$ and $x_1 \in C_1 - C_2$ then there is another circuit containing x_1 contained in $C_1 \cup C_2 - x$

Submodularity of the rank function

- M is a matroid if its rank function is *submodular* :
- for any two subsets, X and Y, of the ground set
  $r(X \cap Y) + r(X \cup Y) = r(X) + r(Y)$
- This is related to the dimension formula for subspaces of a vector space:
- $\dim(U \cap V) + \dim(\mathrm{span}(U \cup V)) = \dim(U) + \dim(V)$
  where U and V are subspaces of a vector space

Alternative definition of transversal matroids

- Suppose the ground set E is the union of *m* sets $A_1, A_2, \ldots A_m$
- The *transversal matroid* induced by these sets can be defined via an E,[*m*] bipartite graph with edges (*e*,*i*) whenever $e \in A_i$
- The independent sets of this matroid are the subsets of E that are saturated by matchings in this bipartite graph

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- Papadimitriou and Steiglitz, Combinatorial Optimization, Dover (1998), chapter 12, pp. 271-306
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