An Efficient Algorithm for Guard Placement in Polygons with Holes

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*Supported in part by DIMACS research assistantship
†Supported in part by National Science Foundation Grants #CCR-88-03549 and #CCR-91-04732.

Abstract

In this paper we consider the problem of placing guards to supervise an art gallery with holes. No gallery with n vertices and h holes requires more than $\lfloor (n+h)/3 \rfloor$ guards. For some galleries this number of guards is necessary. We present an algorithm which places the $\lfloor (n+h)/3 \rfloor$ guards in $O(n^2)$ time.

1 Introduction

Art gallery problems ask the following question: how many guards are always sufficient and sometimes necessary to see every point in an art gallery. The floor plan of the gallery is viewed as a simple polygon. A guard is a point in the polygon. If this point coincides with a vertex, then the guard is called a *vertex guard*. If there is no restriction on the location of the point, then the guard is a *point guard*. We are concerned with the visibility between the guards and the remaining points in the polygon. A guard x sees a point y if the line segment $\overline{xy}$ does not intersect the exterior of the polygon. The original art gallery problem, posed in 1973 by Klee [5], asks for the number of guards required to survey any art gallery with n walls and no interior obstructions. The problem was soon solved by Chvátal [3], who showed that $\lfloor n/3 \rfloor$ vertex guards are both necessary and sufficient. A different and very simple sufficiency proof of the same result was given in 1978 by Fisk [4]. This proof was used by Avis and Toussaint to implement an algorithm running in $O(n \log n)$ time, to assign positions to the guards [1].

If the art gallery is allowed to have obstructions (e.g. pillars) in its interior, the corresponding floor plan is a simple polygon with other simple, disjoint polygons called holes inside. In such a polygon, $\lfloor n/3 \rfloor$ guards are no longer sufficient. In 1984, Shermer showed that for any polygon with one hole, $\lfloor (n+1)/3 \rfloor$ is a tight bound on the number of guards needed. Shermer had already established in 1982 that $\lfloor (n+h)/3 \rfloor$ guards are necessary for a polygon with n vertices and h holes. Shermer's results can be found in [8]. In 1991, two independent and dramatically different proofs were generated for the fact that $\lfloor (n+h)/3 \rfloor$ point guards are also sufficient: one by us [2] and one by Hoffmann, Kaufmann and Kriegel [7]. Neither paper gave details of algorithms for placing the guards, but the algorithm derived from the Hoffmann *et al.* proof has complexity $O(n^3 \log n)$ [6]. In this paper we present an $O(n^2)$ algorithm to place the guards. Since the algorithm derives from the constructive nature of our proof, we start by giving the proof.

Fisk's proof that $\lfloor n/3 \rfloor$ guards are sufficient in any polygon without holes, starts with an arbitrary triangulation of the polygon. The second step is to 3-color the graph of the triangulated polygon, i.e. colors are assigned to the vertices in such a way that no two vertices that share an edge in the graph have the same color and only three

colors are used. In the third step, guards are placed at the vertices of the least used color. At most $\lfloor n/3 \rfloor$ vertices can be colored by this color, so no more than $\lfloor n/3 \rfloor$ guards are used. Finally it remains to note that since all three vertices of a triangle have different colors, every triangle has a guard at one of its vertices. This guard sees every point inside the triangle. Every point of the polygon is inside some triangle and thus the guards see all points in the polygon. To show the sufficiency of $\lfloor (n+1)/3 \rfloor$ guards in any polygon with exactly one hole, Shermer uses an arbitrary triangulation of the polygon with the hole. This triangulation must contain a cycle of triangles, i.e. if we construct the dual of the triangulation with one node for each triangle and an edge between two nodes if the corresponding triangles share an edge, then the dual graph has a cycle surrounding the hole. Shermer first shows that to prove the sufficiency of $\lfloor (n+1)/3 \rfloor$ guards for any triangulation, it is enough to provide a proof for a *reduced triangulation* consisting only of a cycle of triangles perhaps with at most single triangles attached. In some of these triangulations it is not possible to pick $\lfloor (n+1)/3 \rfloor$ guards so that every triangle has a guard at one of its vertices. Shermer calls these configurations *tough triangulations* and makes a case study to show that in each case $\lfloor (n+1)/3 \rfloor$ guards are still sufficient.

Our approach to show that $\lfloor (n+h)/3 \rfloor$ guards are sufficient in any polygon with n vertices and h holes, is radically different: first connect each hole to the exterior with a quadrilateral "channel" and then triangulate the hole-free version of the polygon. The channels are such that (1) only one new vertex is introduced for each channel and (2) a triangle in the remaining polygon sees all of the channel. Our triangulation is not arbitrary. We force these special triangles to be included with the result that a guard placement based on 3-coloring in the hole-free polygon will automatically cover the channels. In section 2 we use this method to give a new proof for the sufficiency of $\lfloor (n+1)/3 \rfloor$ guards when $h = 1$ and in section 3 we extend this result to an arbitrary h. The computational complexity and other aspects of the algorithm is discussed in section 4.

2 Guarding a polygon with one hole

Let P be a polygon with one hole and n vertices. Our procedure for proving that P can be guarded by $\lfloor (n+1)/3 \rfloor$ guards has the following steps. Construct a channel

C that joins the hole to the exterior boundary of P, in such a way that at most one additional vertex is created. Then remove C, producing a new polygon P^1 with $n+1$ vertices and no holes. This polygon can be guarded by $\lfloor (n+1)/3 \rfloor$ guards. The only part of the original polygon left unguarded is the channel C. To ensure that C is also seen by at least one guard, we associate with C during construction a triangle T in the new polygon P^1. T has the property that a guard arbitrarily placed within it can see the entire channel in the original polygon. When P^1 is triangulated to place the guards, we force T to be one of the triangles. After 3-coloring, one of T's vertices is colored by the least used color. The guard placed at this vertex sees all of C and the $\lfloor (n+1)/3 \rfloor$ guards placed can then see not only P^1 but also all of the original polygon P.

We describe three methods of constructing C such that (1) only one new vertex is introduced and (2) a triangle in the remaining hole-free polygon sees all of C. All the methods involve rotations of a ray around a source point. We adopt the convention that if the ray hits two or more vertices simultaneously, then the "first vertex encountered" refers to the vertex closest to the source point. This convention allows us to dispense with the need to assume general position of the vertices. In the following, let the vertices on the exterior boundary B be $p_0, ..., p_{m-1}$ in clockwise order and let the vertices on the hole H be $h_0, ..., h_{n-m-1}$ in counterclockwise order. The first way to construct a channel is to pick an edge $h_{k-1}h_k$, where h_k is a reflex vertex of P on H, and extend it through h_k until it hits the exterior boundary at a point q on edge $p_i p_{i-1}$ (see figure 1). Create a new vertex q' a distance of ϵ away from q in the direction of p_{i-1} and connect q' to h_{k+1}. C is the area h_k, q, q', h_{k+1}. The edge qq' has length ϵ and is called the $\epsilon - side$ of C. The new polygon P^1 has vertices $p_1, ..., p_{i-1}, q'$, $h_{k+1}, ..., h_{k-1}, q, p_i, ..., p_m$, using modular arithmetic. The first property of a channel construction is satisfied because two new vertices q and q' are present, but h_k now lies interior to the edge $h_{k-1}q$ and is no longer a vertex. To satisfy the second property, a triangle T is required. If T can be found such that the area of $C \cup T$ is convex and empty, then every vertex of T is guaranteed to see all of C and the second property of a channel construction is satisfied. We choose as the base of T one side of C. In this construction $q'h_{k+1}$ is that side. A third vertex $v \notin q'h_{k+1}$ to serve as the apex of T must be found. Choose v as the first vertex encountered when $h_{k+1}q$ rotates around h_{k+1} with q sliding along its edge towards p_{i-1}. If the

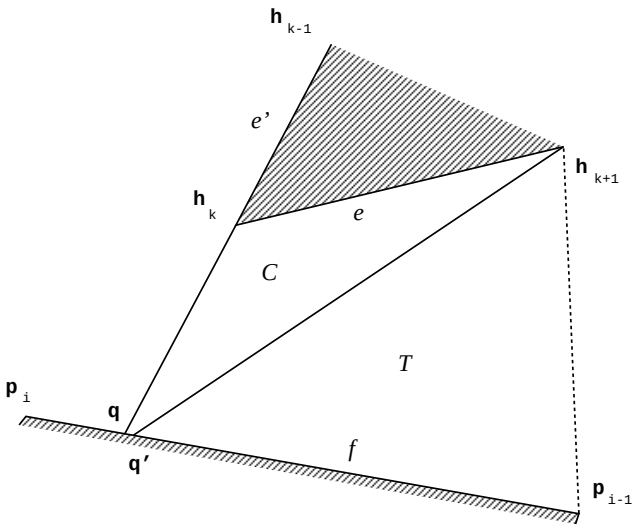

Figure 1: Edge $h_{k-1}h_k$ is extended through h_k.

quadrilateral $Q = h_{k+1}, h_k, q, v$ is convex and empty then area $C \cup T$ is guaranteed to be convex and empty. Above we extended edge $h_{k-1}h_k$ through h_k. A symmetric construction can be produced if we instead extend edge $h_{k+1}h_k$ through h_k.

In case 2.1 we formalize the conditions that must hold for this type of construction to be possible. We let h_l represent a vertex adjacent to h_k in either direction. In the *forward* version of case 2.1, h_l lies in a counterclockwise direction from h_k and in the *backward* version of case 2.1, h_l lies in a clockwise direction from h_k. We consolidate the two versions into one case:

Case 2.1 *There exists a reflex vertex h_k of P on H with incident edges $e = h_kh_l$ and e' and an edge $f = p_ip_j$ on B, such that the following conditions hold:*

- *The extension of e' through h_k hits edge f on B at $q \neq p_j$.*
- *A half-plane defined by $\overline{h_kq}$ contains both e and p_j.*
- *There exists a vertex v on H or B and a quadrilateral $Q = h_l, h_k, q, v$ such that*
  1. *$v \notin \overline{qh_l}$*
  2. *v is the first vertex encountered by h_lq as q slides along f*
  3. *Q is convex*
  4. *Q has an empty interior*

Note that it is possible within case 2.1 for the extension from h_k to touch a vertex on B. In the special case where q coincides with a vertex p_i and h_k, q and p_{i-1} do

not form a right turn, the construction results in a polygon P^1 with n vertices. If the extension from h_k first touches B at a vertex p_j and h_k, p_j and p_{j-1} form a right turn then the extension is continued. The removal of C divides P into pieces. These pieces are triangulated separately and then joined at p_j before the triangulation graph is 3-colored.

The second method of constructing a channel is to connect vertex h_k directly to p_i, create vertex p_i' a distance of ϵ away from p_i in the direction of p_{i-1} and connect it to h_{k+1} (see figure 2). C is the area h_k, p_i, p_i', h_{k+1} with ϵ-side $p_i p_i'$. The base of triangle T

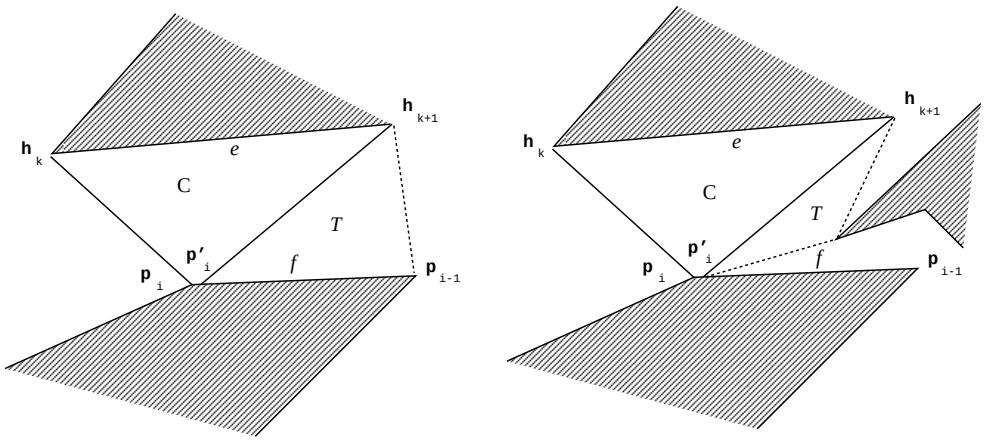

Figure 2: Vertex h_k is connected directly to vertex p_i.

lies along the side $p_i' h_{k+1}$ and the third vertex v is picked so that $Q = h_{k+1}, h_k, p_i, v$ is empty and convex. A backward version of the construction ensues if we instead place vertex p_i' in the direction of p_{i+1} and connect it to h_{k-1}. This type of construction is applicable in the following case:

Case 2.2 *There exist an edge $e = h_k h_l$ on H and an edge $f = p_i p_j$ on B such that the following conditions hold:*

- *A half-plane defined by $\overline{h_k p_i}$ contains both e and p_j.*
- *There exists a vertex v on H or B and a quadrilateral $Q = h_l, h_k, p_i, v$ such that*
  1. *$v \notin \overline{p_i h_l}$*
  2. *v is the first vertex encountered by $h_l p_i$ as p_i slides along f.*
  3. *Q is convex*
  4. *Q has empty interior*

For the third and final method of constructing C, we extend an edge from a vertex

p_i on the exterior boundary until it hits an edge e on H at a point k (see figure 3). Create a new vertex p_i' next to p_i and connect it to the endpoint of e. C has ϵ-side $p_i p_i'$. A forward or backward version of the construction is produced depending on which incident edge is extended from p_i. This type of construction is applicable in the following case:

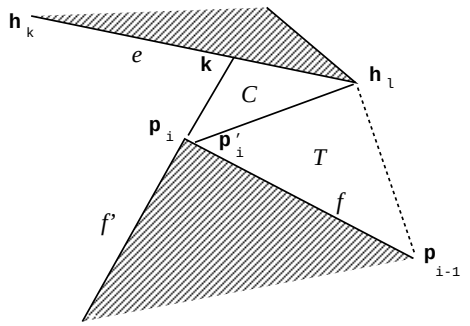

Figure 3: Edge $p_{i-1}p_i$ is extended through p_i.

Case 2.3 *There exists an edge $e = h_k h_l$ on H and a reflex vertex p_i with incident edges f and f' on B, such that the following conditions hold:*

- *The extension of f' through p_i hits e at $k \neq h_l$.*
- *A half-plane defined by $\overline{p_i k}$ contains both f and h_l.*
- *There exists a vertex v on H or B and a quadrilateral $Q = h_l, k, p_i, v$ such that*
  1. *$v \notin \overline{p_i h_l}$*
  2. *v is the first vertex encountered by $h_l p_i$ as p_i slides along f*
  3. *Q is convex*
  4. *Q has an empty interior*

Each one of the above cases was specified to allow a corresponding type of channel to be constructed between an edge on H and an edge on B. The ϵ-side of C always lies on B.

Lemma 2.1 *If one of the cases above holds for a polygon P with one hole, then a channel can be removed from P in such a way that the new polygon P^1 has:*

- *$n+1$ vertices.*
- *no hole.*
- *a triangle from whose vertices the area of the removed channel is visible in P.*

Proof: In each case the construction adds one vertex to the total, thus the number of vertices is $n+1$. Each constructed channel has one side along the hole and an ϵ-side along the exterior boundary. Thus no hole exists when the channel is removed. One side of T is shared by C, guaranteeing that the two endpoints see all of C. The following method finds the third vertex v of T. Rotate the edge shared by C and T around the endpoint situated on H. Let the other endpoint slide along the boundary edge on which it is situated. The first vertex encountered by the rotating edge is chosen as v. This vertex could be the endpoint of the boundary edge (see figure 1) or some other vertex (see figure 4). The condition that Q must be convex guarantees

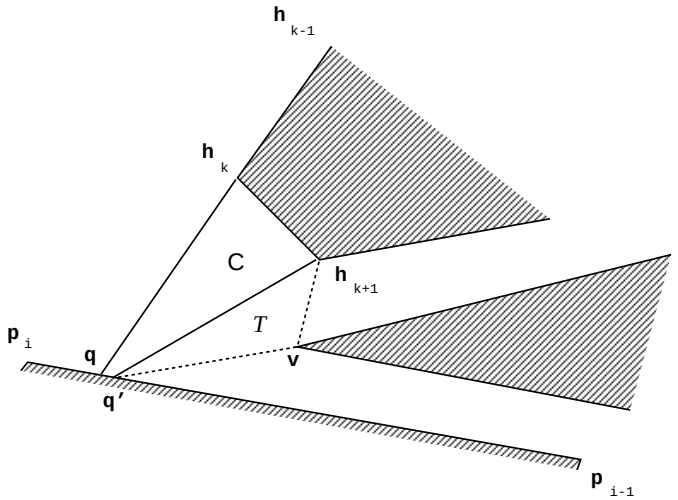

Figure 4: The first vertex encountered as $h_{k+1}q'$ rotates towards an extension of h_kh_{k+1} is chosen as v.

that a vertex is encountered by the rotating edge before the rotation is carried past an extension of the edge shared by C and H (see figure 5). Thus the third vertex of

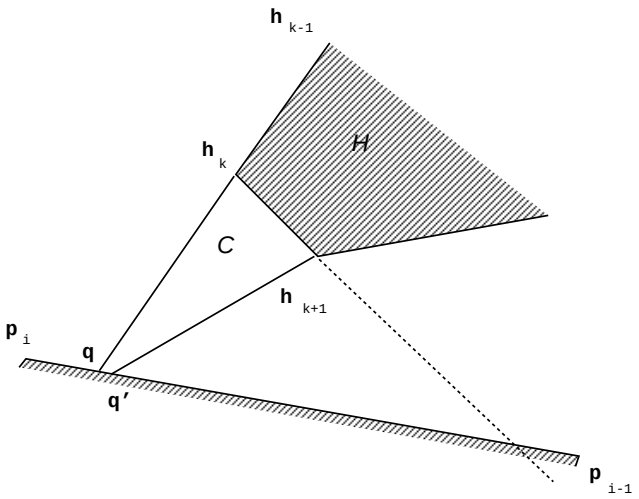

Figure 5: If a vertex v is not found before edge $h_{k+1}q'$ has rotated past the extension of h_kh_{k+1} through h_{k+1} then Q is not convex.

T sees all of C. □

Lemma 2.1 guarantees that if one of the cases 2.1, 2.2 or 2.3 holds then a channel with the required properties can be constructed. To prove that any hole can be connected to the exterior by a channel, it remains to show that in every polygon with one hole there exist two edges (one on the hole and one on the exterior boundary) such that one of the cases is applicable. Before proceeding to this proof, it is helpful to understand what happens when none of the cases applies to two specified edges, in particular when vertex v as specified in the conditions for cases 2.1, 2.2 or 2.3, does not exist.

We begin with the forward version of case 2.1. Let h and p represent h_k and q respectively and assume that the first two conditions of case 2.1 holds with e incident to h in the counterclockwise direction. For case 2.1 not to apply, it must be true that the requisite vertex v does not exist. Let l_h be a line through e and let l_p be a line through f (see figure 6). If l_h and l_p are parallel or if they intersect at a point z on the opposite side of hp from h_l then quadrilateral $Q' = h_l, h, p, p_j$ is convex (see figure 6a). In this situation, one or more vertices must lie inside Q', or Q' would be both empty and convex. At least one of these vertices must also lie in the triangle h, p, h_l, since otherwise one vertex v could be picked from among the vertices in Q' in such a way that h_l, h, p, v is empty and convex and $v \notin \overline{ph_l}$. If instead z and h_l lie on

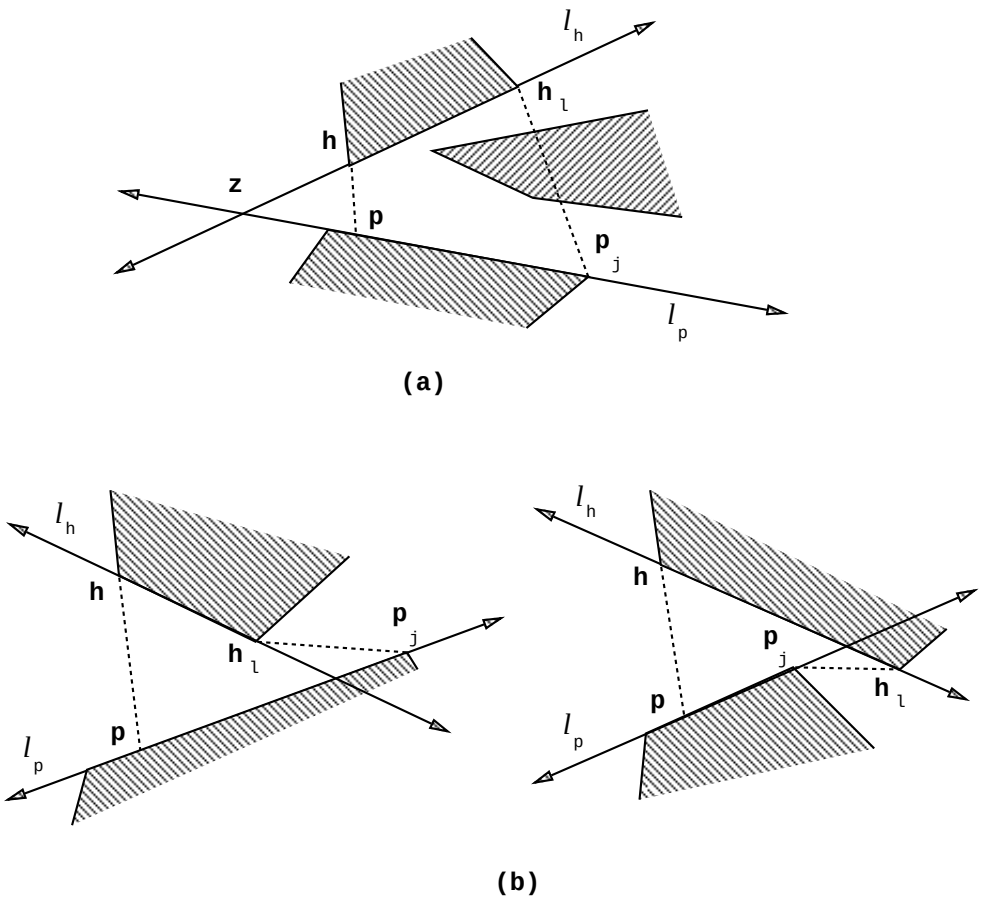

Figure 6: (a) Q' is nonempty. (b) Q' is nonconvex.

the same side of hp, then Q' may still be nonempty but it is also possible that Q' is empty and nonconvex (see figure 6b). In Q', the interior angles at both h and p are convex since the first conditions of case 2.1 was assumed true. Either h_l or p_j must then be a reflex vertex.

We conclude from the preceding paragraph that for the forward version of case 2.1 to fail either some vertex lies in triangle h, p, h_l or Q' has a reflex angle at h_l or p_j. The forward versions of cases 2.2 and 2.3 fail in the same situations: in case 2.2, h and p represent two vertices h_k and p_i, chosen such that the first condition of the case holds; in case 2.3, h and p represent k and p_i such that the first two conditions of the case hold.

We now assess the feasibility of the backward versions of cases 2.1, 2.2, and 2.3, where the forward versions are already known to fail. This is more complicated because we do not make any assumptions about the characteristics of h_l and p_j and the reasons for failure need not involve the existence of an empty convex quadrilateral Q. Therefore, we determine what the situation should be for one of the backward versions to apply. Specifically, if Q' as described above is empty and nonconvex with a reflex angle at h_l then if an extension of the counterclockwise edge through h_l hits p, p_j there exists a vertex v such that $Q = h, h_l, q, v$ is convex and empty and case 2.1 applies (see figure 7a). If Q' is empty and nonconvex with a reflex angle at p_j (see figure 7b) then if an extension of the counterclockwise edge from p_j hits e at point k on hh_l, there exists a vertex v such that $Q = h, k, p_j, v$ is convex and empty and case 2.3 applies. Finally if Q is convex and nonempty (see figure 7c) and an extension of the counterclockwise edge from p_j hits e at point k on hh_l, then case 2.3 applies if none of the vertices inside Q lie in the triangle p_j, k, h.

We are now prepared to prove the following lemma:

Lemma 2.2 *At least one of cases 2.1, 2.2 and 2.3 applies to any polygon with one hole.*

Proof: The format of the proof is an algorithm which finds an edge e on H for which one of the cases applies. The strategy of the algorithm is to isolate two reflex vertices a and b of P on H with two corresponding points r and s on B in such a way that one of the cases will apply in the area of P which lies between the two segments

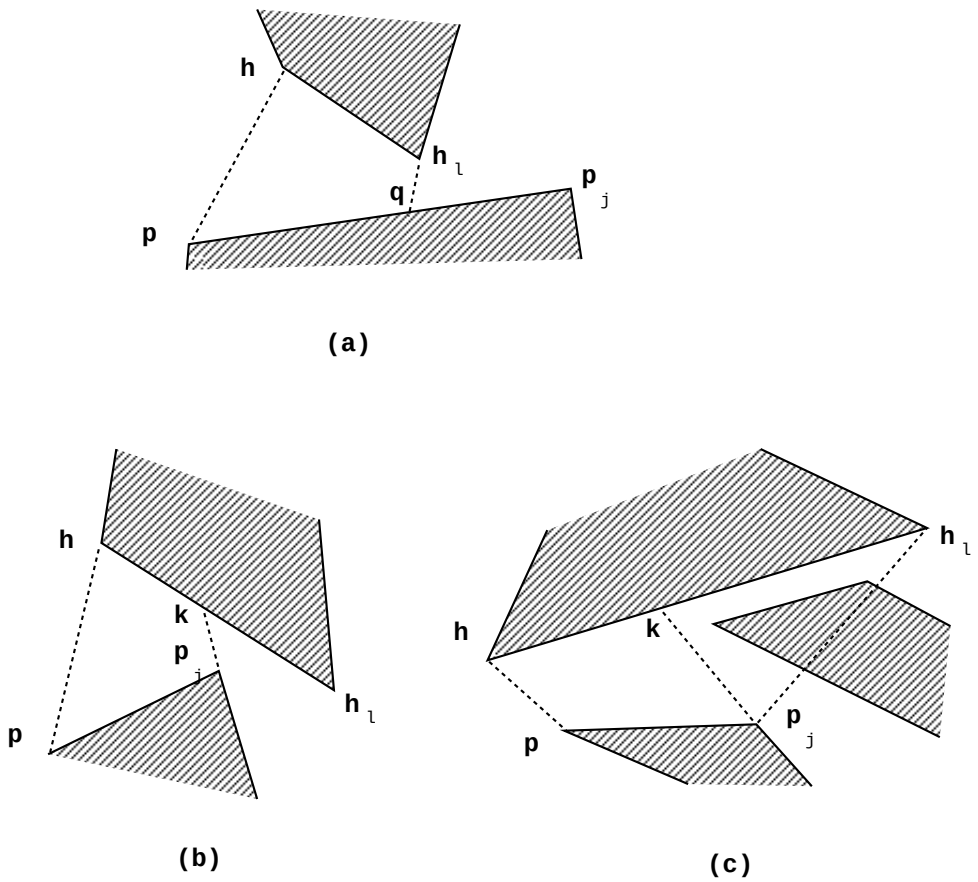

Figure 7: (a) Case 2.1.1 applies. (b) Case 2.1.3 applies.

ar and bs (see figure 8): a must be chosen so that the extension of the edge clockwise

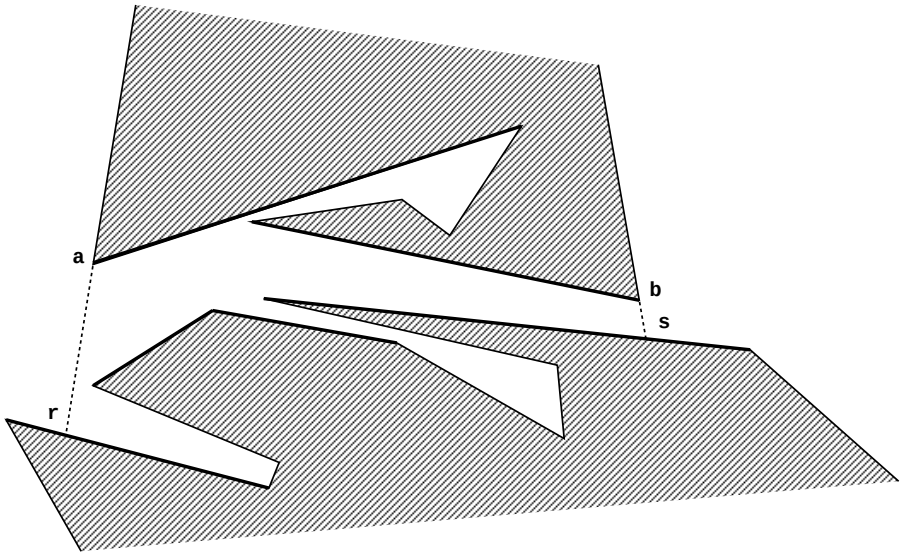

Figure 8: The area between ar and bs is searched for an applicable case.

from a through a intersects B without intersecting any other part of H; b must be chosen so that the extension of the edge counterclockwise from b through b intersects B without intersecting any other part of H and at least one vertex lies on B between r and s. To this end, consider the *relative convex hull* of H given B, defined as the minimum perimeter polygon containing H constrained to lie in B. The extension of an edge from a vertex which lies both on H and the relative convex hull of H given B cannot intersect H. Let a be the leftmost such vertex and let r be the first intersection point between B and the extension from a, such that a, r and p_{i-1} form a left turn, where $p_i p_{i-1}$ is the edge on which r lies. To locate b, traverse the vertices of the relative convex hull of H given B in a counterclockwise direction from a and pick as b the first vertex on H such that an extension of the edge counterclockwise of b through b intersects B at a point s without intersecting the segment ap_{i-1}. Vertex b must exist. We walk around the relative convex hull of H given B and stop to check every vertex which lies on H. An extension of the counterclockwise edge from such a vertex will not hit H. However, we must also make sure that it does no intersect ap_{i-1}. Let the direction of $\vec{ar}$ be d and let the direction of $\vec{rp_{i-1}}$ be d'. Since a, r and p_{i-1} form a left turn, it follows that $d' < d + 180$. If we start the traversal at a facing in direction d and if we complete the traversal and return to a then we will

have turned 360 degrees. Consequently we must at some vertex w be facing in the direction d'' where $d'' > d' + 180$ or we could never return to a. An extension of the edge counterclockwise from w through w cannot intersect ap_{i-1}. Segment bs lies in a counterclockwise direction from ar.

Having delimited the search area, we start the search by placing a segment $h'p'$ so that h' coincides with a and p' coincides with r. The segment $h'p'$ will be walked along the two chains in steps. This move is always in a counterclockwise direction with $h'p'$ growing and shrinking to keep one endpoint on each chain. At each step, check to see whether any case applies. If no case applies, then either h' or p' moves to a new vertex. The move guarantees that h' and p' see each other at all times. Specifically, let e be the edge incident to h' in a counterclockwise direction and let f represent $p'p_{i-1}$. First try to apply case 2.1 and 2.3 using edges e and f. If unsuccessful, then one of the following situations holds: (1) Q is nonempty; (2) Q is empty with a reflex angle at h_l; or (3) Q is empty with a reflex vertex at p_j. In either case, we wish to move one endpoint of $h'p'$ and try again. To decide what the particular move should be, we first rotate $h'p'$ around p', letting h' slide along e in a counterclockwise direction. Let x be the first vertex encountered by the rotating $h'p'$. Vertex x will either be found before the endpoint of e has been reached or x will be this endpoint. If x lies on B, we take a step by moving p' to x, leaving h' in its old position (see figure 9). If x lies on H, then h' is moved to x and p' remains unchanged (see figures 10 and 11).

Note that the choice of x ensures that x sees both h' and p' before the move and consequently after the move h' and p' see each other. The choice of x also implies that if x lies on B then x is a reflex vertex.

After the move, we try again to apply the cases in the following order:

- Try the forward version of case 2.2 if $p' \neq r$ and construct a channel if possible.
- Try the forward version of case 2.1 if $p' = r$, using $q = p'$, but do not construct a channel. While $p' = r$, it is possible that case 2.2 does not apply because $h'p_i$ may be obstructed (see figure 12) and no nonempty quadrilateral Q exists. Case 2.2 is used to find a vertex x to which h' or p' will be moved in the next step. If no vertex v exists such that quadrilateral $Q = h_l, h', p', v$ is convex and empty, then x is found by the rotation of $h'p'$ as above. If vertex v exists and

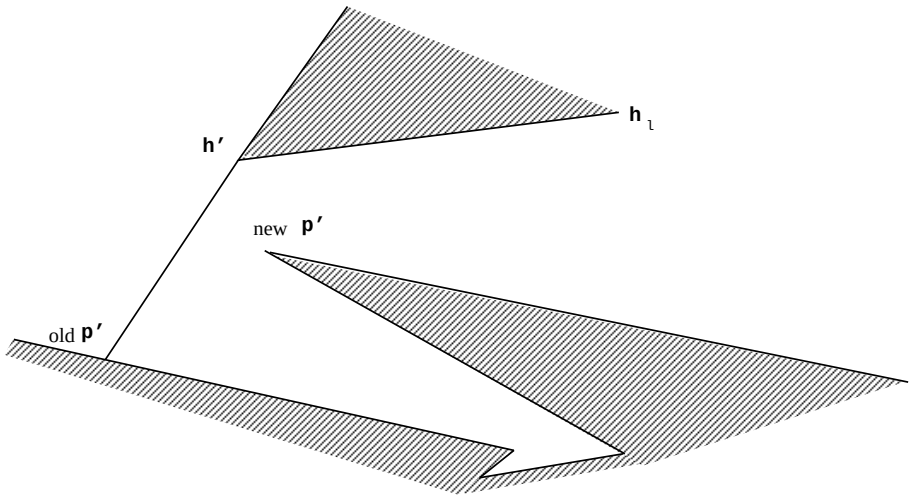

Figure 9: The first encountered vertex as $h'p'$ rotates lies on the exterior boundary. This vertex becomes the new p'.

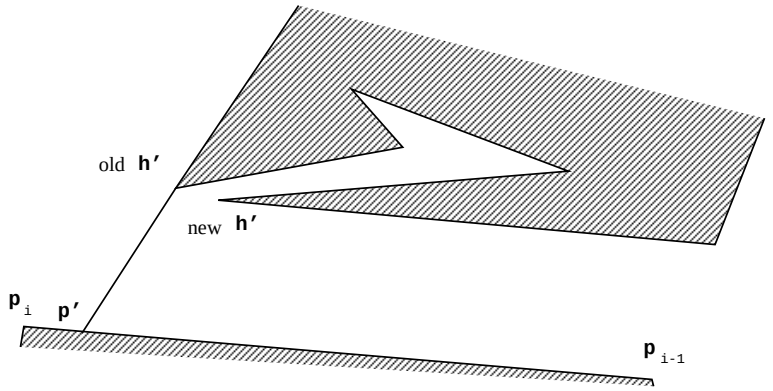

Figure 10: The first encountered vertex is a hole vertex. It becomes the new h'.

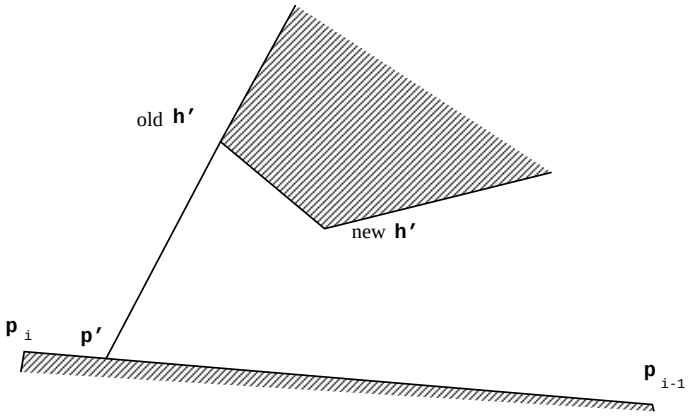

Figure 11: The endpoint of e becomes the new h'

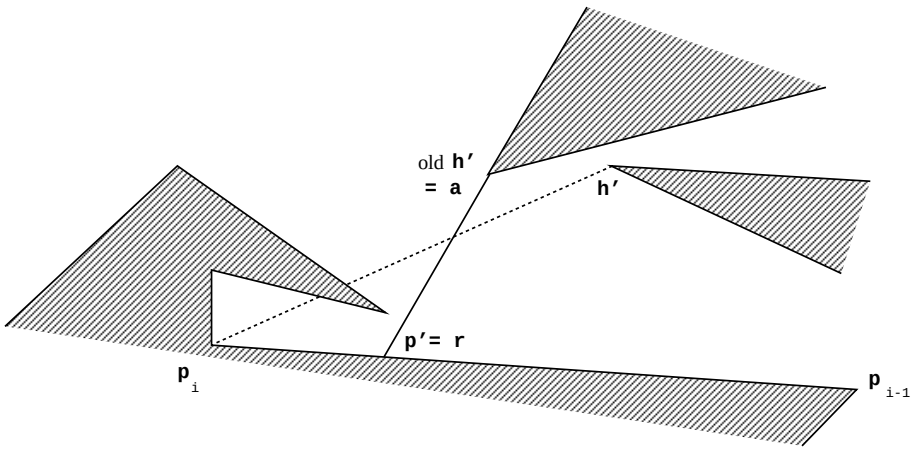

Figure 12: While $p' = r$ it is possible that h' does not see p_i.

$v \neq p_{i-1}$, then v is the vertex to which h' or p' will be moved in the next step. Finally, if $v = p_{i-1}$, then the backward version of case 2.2 will apply when tried below and no further move is needed for $h'p'$.

- Try the backward version of case 2.1 if $x = h_l$ and the extension of e' hits f and construct a channel if possible.
- Try the backward version of case 2.2 if $v = p_{i-1}$ and construct a channel if possible.
- Try the forward version of case 2.3 if p' is a vertex and construct a channel if possible.
- Try the backward version of case 2.3 with an extension of the edge counterclockwise from p' if p' is a vertex and construct a channel if possible.
- Try the backward version of case 2.3 and construct a channel if possible.

If none of the above steps was successful, then either some vertex x is present in the triangle h', p', h_l or in the quadrilateral h_l, h', p', p_{i-1}. We again use rotations of $h'p'$ to find x and move either h' or p' one step in a counterclockwise direction. Each time we fail to find an applicable case, we move one step closer to the segment bs until h' is located at the vertex adjacent to b and p' is the endpoint of the edge on which s lies. At this point the quadrilateral h', p', s, b is convex and empty, since b

and c are chosen so that no part of P intersects bs. Our choice of b and s guarantees that case 2.1 applies from b with $v = p'$ (see figure 13) and the algorithm terminates successfully. □

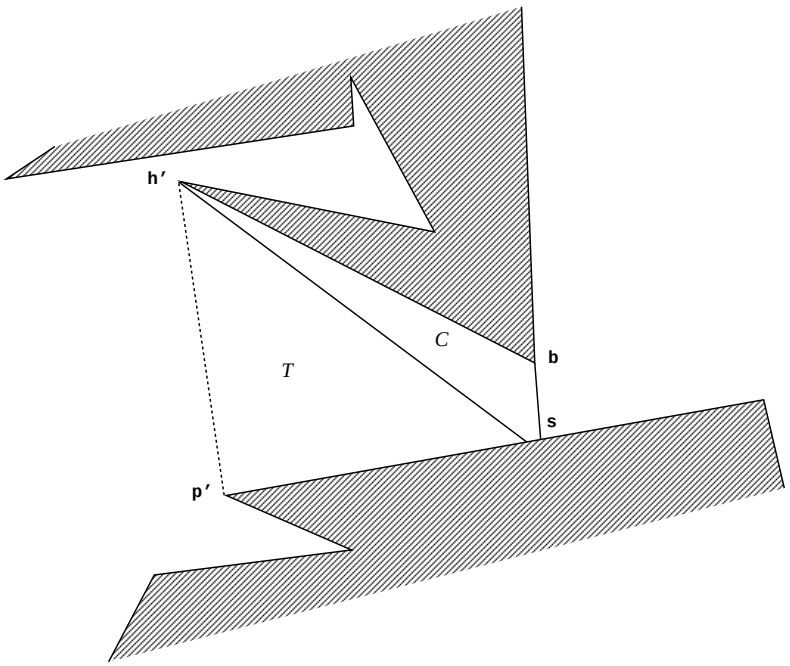

Figure 13: Case 2.1.1 applies.

The algorithm is illustrated in figure 14, where we show each step taken by segment $h'p'$ in the search area from figure 8.

Lemmas 2.1 and 2.2 together prove the following theorem:

Theorem 2.3 *In any polygon P with n vertices and one hole a channel can be removed in such a way that the remaining polygon has:*

- $n + 1$ *vertices*
- *no hole*
- *a triangle from whose vertices the area of the removed channel is visible in P.*

The $\lfloor (n+1)/3 \rfloor$ bound on the number of guards required in a polygon with one hole follows from the theorem. While all the guards are vertex guards in P^s, not all these vertices exist in P. The guards in P are therefore point guards.

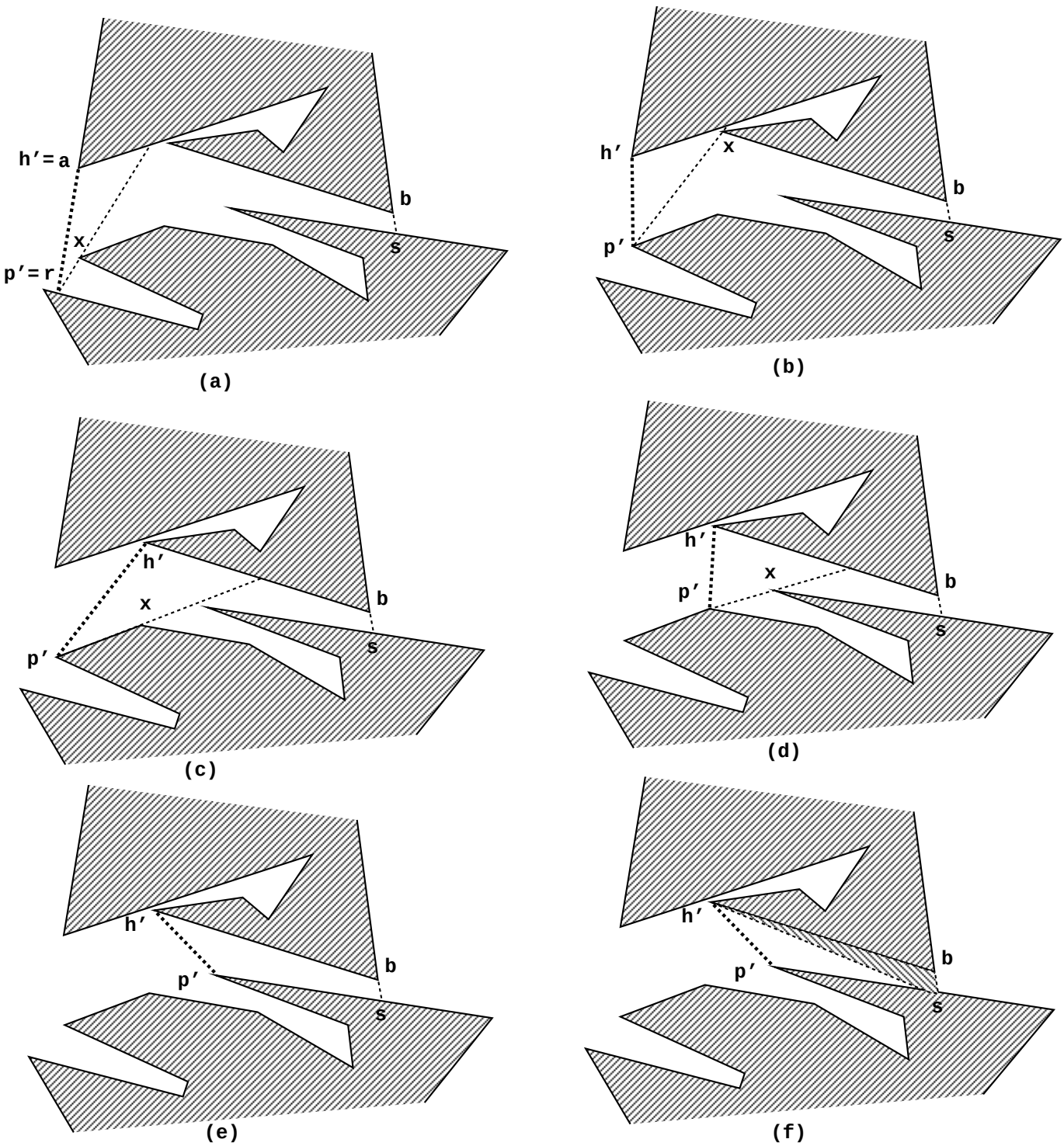

Figure 14: The steps taken by $h'p'$ before an applicable case is found. The heavy dashed line indicates the current position of $h'p'$. In (a), (b), (c) and (d) the thin dashed line shows how far $h'p'$ must rotate before a vertex x is encountered. In (e) the quadrilateral h', b, s, p' is convex and empty, case 2.1.1 applies and the resulting channel is seen in (f).

We conclude this section by showing how our algorithm handles a polygon, whose triangulation is tough. As discussed in section 1, a tough triangulation is one where one triangle must be left without a guard at one of its vertices in order not to use more than $\lfloor (n+1)/3 \rfloor$ guards. Figure 15a shows such a polygon with ten vertices. If we require every triangle to have a guard at one of its corners, i.e. if we require the guards to be *combinatorial*, then four guards are needed. Figure 15b shows the channel constructed for this polygon by our algorithm and also the resulting guard assignment. The removal of a channel from the polygon has the result that the

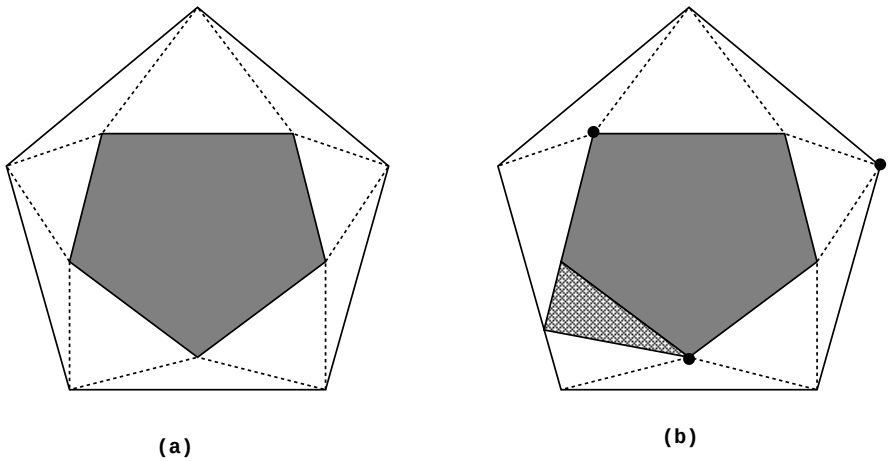

Figure 15: (a) Triangulation where 4 combinatorial guards are necessary. (b) After the indicated channel is removed, guards are placed at the black vertices.

triangulation of the new polygon contains one less triangle. The assigned guards are combinatorial in the hole-free polygon, but not in the original polygon. The problem of making sure that one triangle is guarded without having a guard at one of its vertices is handled automatically by the algorithm and does not present any problems.

3 Guarding a polygon with an arbitrary number of holes.

When the polygon has h holes, the construction of a channel as in section 2 above, will be repeated h times. Each construction and removal of a channel results in a new polygon with one additional vertex and one less hole. After h repetitions we have a

polygon with $n+h$ vertices and no holes. Inside this polygon there will exist a triangle for each channel such that the vertices of the triangle see all of the area covered by the corresponding channel in the original polygon. In the remainder of this section we show exactly how the triangles and constructions are modified to ensure that these triangles exist. With the triangles in place, a guard assignment based on 3-coloring in the hole-free polygon will cover the original polygon with holes. Since the new polygon has $n+h$ vertices, the number of guards is $\lfloor (n+h)/3 \rfloor$. These guards are vertex guards in the hole-free polygon, but point guards in the original polygon, since new vertices were added during the channel constructions.

To construct the first channel, we start the search at vertex a of one hole and point r, which satisfy the same conditions as in the case of $h = 1$. We place the segment $h'p'$ to coincide with ar and try to construct a channel. If no case applies, either h' or p' is moved in a counterclockwise direction as described in section 2, but now h' may move to a different hole. This substitution does not affect the existence of an applicable construction; as long as no construction is found we keep moving h' or p' and eventually must reach a situation where the extension of the edge counterclockwise from a vertex b through b hits the boundary at s and the quadrilateral h', b, s, p' is convex with an empty interior. At this point case 2.1 applies. The vertex b is not predetermined; instead we just test each new vertex adjacent to h' in a counterclockwise direction as a potential b. The reason for this change in procedure is that with several holes, the exterior boundary may partially hide some holes from each other and as we move in a counter-clockwise direction from a, even some holes on the relative convex hull may be skipped (see figure 16). At the point where a construction is found, one hole (not necessarily the hole where the search started) is joined to the boundary by a channel adding one vertex to the total. We then proceed to search for the next channel, starting at the point where we last switched from one hole to another, or, if no such switch occurred, we restart the search at a newly picked vertex a. Let the channels be $C_1, C_2, ..., C_h$. The construction and removal of the first channel C_1 results in a polygon P^1 with $n+1$ vertices and $h-1$ holes. Constructing and removing channel C_2 results in polygon P^2 with $n+2$ vertices and $h-2$ holes, etc., until we have a polygon P^h with $n+h$ vertices and no holes. We must at this point also have h triangles $T_1, T_2, ..., T_h$ inside P^h such that guards anywhere inside these triangles can see all of the area covered by channels $C_1, ..., C_h$

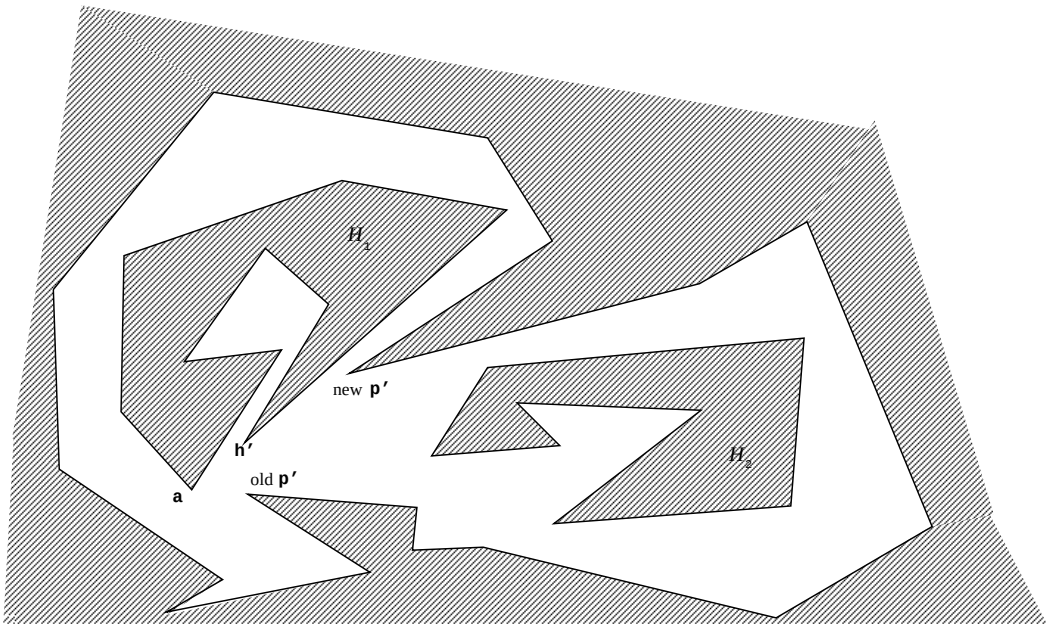

Figure 16: No vertex on hole H_2 becomes h' since part of B inserts itself between H_1 and H_2.

in the original polygon P.

Our main concern with multiple holes is that once we have a completed channel and its associated triangle, the construction of a subsequent channel may interfere with the existing triangle. Let t_1, t_2 and t_3 be the vertices of triangle T_r associated with channel C_r. The side t_1t_2 is the side shared with C_r and t_1 was originally a hole vertex. Assume a channel C_s $(s > r)$ is constructed. After the construction of C_s the following two conditions must hold:

1. Each vertex t_1, t_2 and t_3, if superimposed on the original polygon P, sees the area covered by C_r.

2. T_r must exist within P^s (and ultimately within P^h).

A triangle T_r may go through multiple changes during the course of the algorithm. Here we detail the types of changes that can occur as well as additional restrictions on some of the cases to guarantee that conditions (1) and (2) still hold at the end of the channel constructions. A triangle T_r has one of two types. Either t_2t_3 is an edge on the exterior boundary forming an angle of less than 180 degrees with t_1t_2 (see figure 17), or t_2t_3 is interior to P^r, in which case another boundary edge t_2p_j also

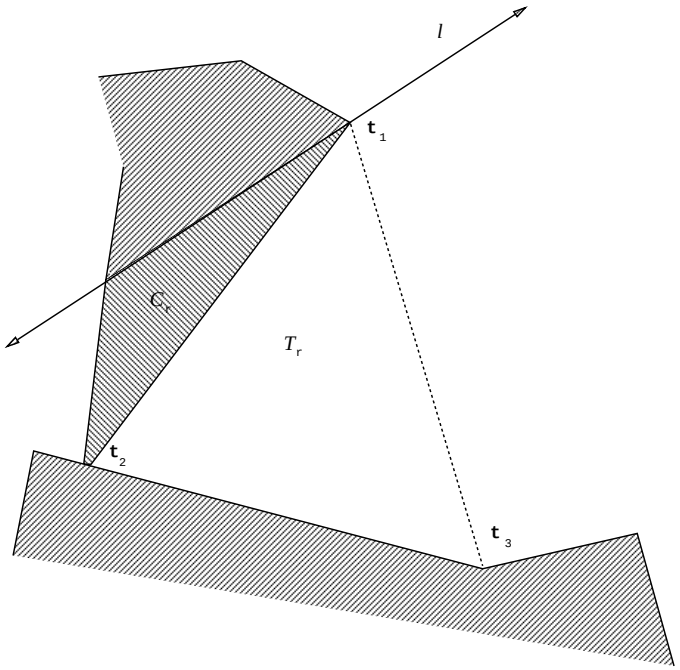

Figure 17: The side t_2t_3 lies on the exterior boundary.

forming an angle of less than 180 degrees with t_1t_2 originates at t_2 (see figure 18). For condition 1 to be violated, one of the vertices of T_r must have moved. Vertices are moved only in constructions where cases 2.1 and 2.3 are applied. The construction used in case 2.1 extends an edge from a reflex vertex on H and the construction in case 2.3 extends an edge from a reflex vertex on B. Vertex t_3 is the only vertex of T_r that can lie on a hole. Thus if case 2.1 was applied to construct C_s then vertex t_3 on H was moved. If case 2.3 was applied then vertex t_1 or t_3 (if t_3 lies on B) was moved. Vertex t_2 is never moved since t_2 is not a reflex vertex. For condition 2 to be violated, either a vertex has moved as described above or C_s removed part of T_r or both. C_s has removed part of T_r if the ϵ-side of C_s is placed along edge t_1t_2 or t_2t_3 (or t_2p_j in the second version of T_r). In the first version of T_r an ϵ-side along edge t_1t_2 will cut off vertex t_1 from the rest of T_r and an ϵ-side along t_2t_3 will cut off vertex t_3. In the second version of T_r the vertices cut off are t_1 or t_2 in the first case and t_3 in the second case.

There are thus six instances in which the construction of C_s interferes with the conditions that must hold for T_r:

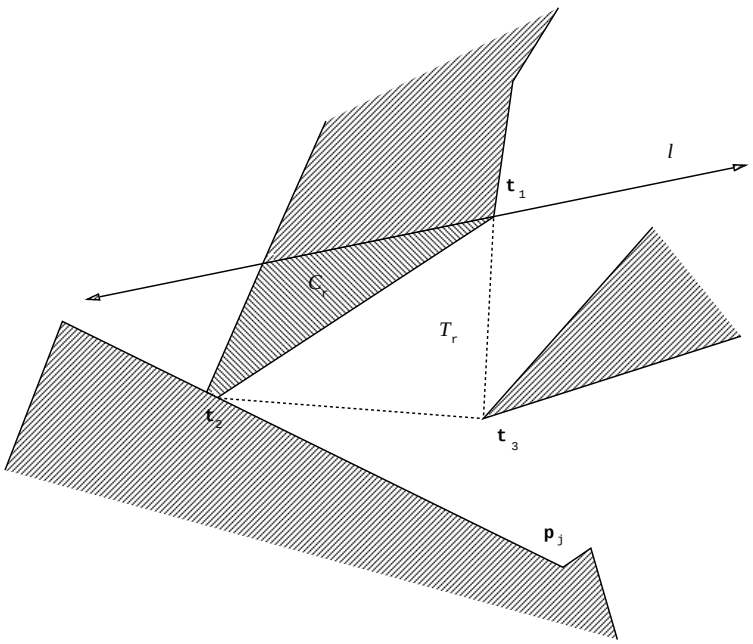

Figure 18: The side t_2t_3 is interior to the polygon.

- Vertex t_3 lies on B and is moved.
- Vertex t_3 lies on H and is moved.
- Vertex t_1 is moved.
- Vertex t_3 is cut off.
- Vertex t_2 is cut off.
- Vertex t_1 is cut off.

We treat each instance as a separate case. Let $Q_r = g_r, h_r, p_r, t_3$ be the quadrilateral that was specified as convex and empty when C_r was constructed. Similarly let $Q_s = g_s, h_s, p_s, v_s$ be the corresponding quadrilateral for C_s. Let l_r and l_s be lines through $g_r h_r$ and $g_s h_s$, respectively.

Instance 1 *The triangle side t_2t_3 is an edge on the boundary. Vertex t_3 is moved.*

Case 2.3 is applied to construct C_s with the result that Q_r becomes nonconvex or nonempty. When t_2t_3 is extended, t_3 moves away from its original position and the

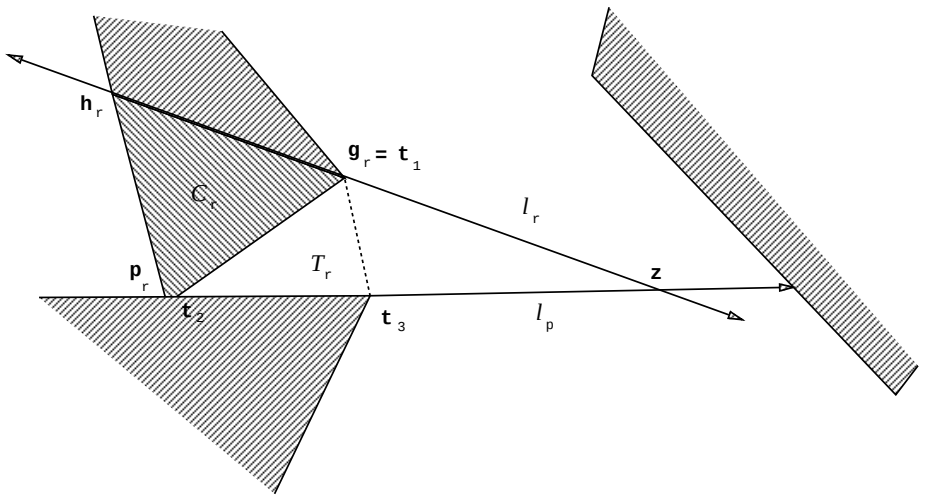

Figure 19: Extending edge t_2t_3 so that t_3 ends up on the opposite side of l makes Q_r nonconvex.

side g_rt_3 rotates around g_r (see figure 19). If Q_r becomes nonempty while still being convex, then there exists some vertex v in Q_r such that $Q'_r = g_r, h_r, p_r, v$ is empty and convex. Vertex v replaces t_3, and a modified version of T_r exists. If Q_r should remain empty but become nonconvex, then the case that was applied to construct C_r no longer applies. Q_r becomes nonconvex when t_3 reaches point z, which is the intersection point between l_r and a line l_p through t_2t_3. Consequently, Q_r may become nonconvex if z and t_3 lie on the same side of h_rp_r. To prevent this problem, we further restrict the cases described in section 2. In particular, we add the condition that, if t_3 and z lie on the same side of h_rp_r and t_3 is chosen as the endpoint of the edge on which p_r lies then, the triangle g_rt_3z must not be empty. An applicable case still always exists since when the quadrilateral g_r, h_r, p_r, t_3 is empty and convex and z lies on same side of h_rp_r as t_3 then a vertex v exists such that $Q'_r = h_rg_rt_3v$ is empty and convex and case 2.2 applies with g_r as h_k and t_3 as p_i.$\diamond$

Instance 2 *The triangle side t_2t_3 is interior to the polygon. Vertex t_3 is moved.*

Case 2.1 is applied to construct C_s with the result that Q_r changes shape. Assume first that the extension from t_3 hits edge t_2p_j. This edge will now have two new vertices: the relocated t_3 (called q in case 2.1) and q' which lies at a distance of ϵ away from q. If t_2 and q' are on opposite sides of q (see figure 20), then $Q_r = g_r, h_r, p_r, t_3$ remains empty and convex with t_3 in its new location, i.e. $t_3 = q$. If t_2 and q' are on

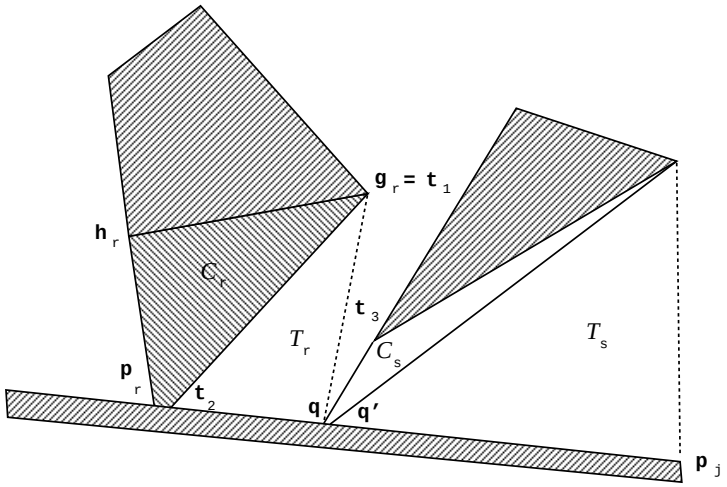

Figure 20: The side clockwise from t_3 is extended through t_3, hitting edge t_2p_j.

the same side of q (see figure 21) then $Q'_r = g_r, h_r, p_r, q'$ is empty and convex. Vertex

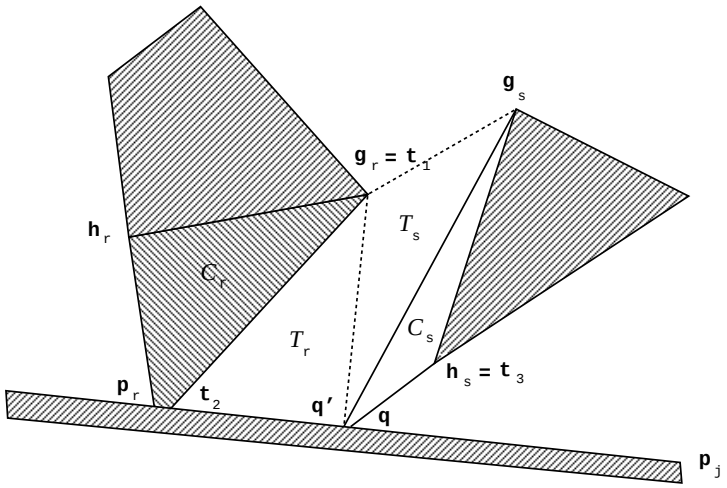

Figure 21: The side counterclockwise from t_3 is extended through t_3, hitting edge t_2p_j.

q' replaces t_3 and T_r exists in P_s in a modified form. The quadrilateral g_s, h_s, p_s, t_1 is convex, since t_3 is the first vertex encountered as g_rp_r rotates around g_r. If it is not empty, then some vertex v_s inside it can be picked such that $Q_s = g_s, h_s, q, v_s$ is empty and convex.

Assume instead that one or both extensions through t_3 hit edge t_1t_2 (see figure 22) and we apply the construction associated with case 2.1. The result is that T_r no longer exists in P^s. There are two possibilities. Either we just started the search

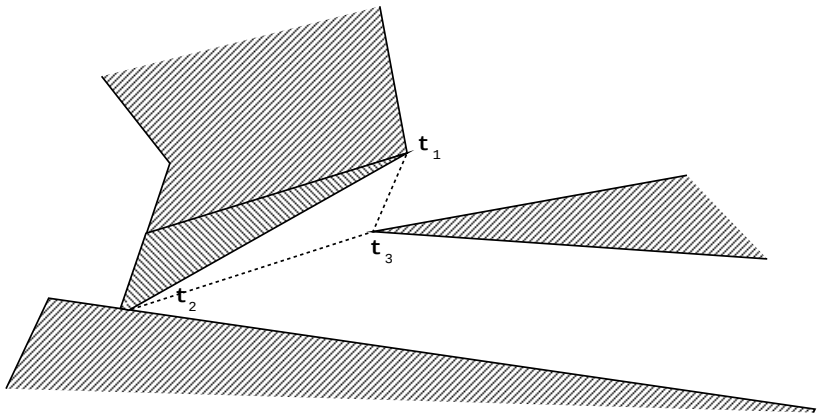

Figure 22: An edge is extended through t_3, hitting side t_1t_2.

for an applicable case with t_3 chosen as a and point r situated on t_1t_2, or the search started at a vertex in a clockwise direction from t_3. If our search just started then our moving line segment $h'p'$ currently has $h' = a$ and $p' = r$. We take a step by moving p' to t_2 and continue the search. The move is in a counterclockwise direction and the new p' sees both h' and r. Since t_3 can not play the role of both a and b, an applicable case is still guaranteed to exist.

If instead our search started at some vertex in a clockwise direction from t_3, then segment $h'p'$ is currently positioned with h' at vertex g_s and p' at vertex t_1. Let l_p be a line through the edge incident to t_1 in a clockwise direction. If l_p intersects t_2t_3 or if t_1 is convex then an empty convex quadrilateral g_s, t_3, t_1, v exists and the construction associated with case 2.2, with edge g_st_3 as e, applied in a previous step (see figure 23). Thus we must have moved forward in our search only if l_p does not intersect t_2t_3 and t_1 is reflex. In this case, no easy modification of T_r exists, and t_3 may be designated b so that we can not move forward. We prevent this problem by disallowing the construction of C_r and T_r in the first place, if it would produce these consequences. Adding the following condition to the cases in section 2 suffices: vertex v_r on H may not be such that the extension of one of its incident edges hits p_rg_r unless vertex g_r is convex or the extension of the edge clockwise from g_r through g_r intersects p_rv_r. The addition of this condition does not change the fact that one of the cases will eventually apply, since if the condition does not hold, the extension of the counterclockwise edge from g_r hits H and g_r cannot serve as b. The next step taken by segment $h'p'$ is to move h' to v_r. $\diamond$

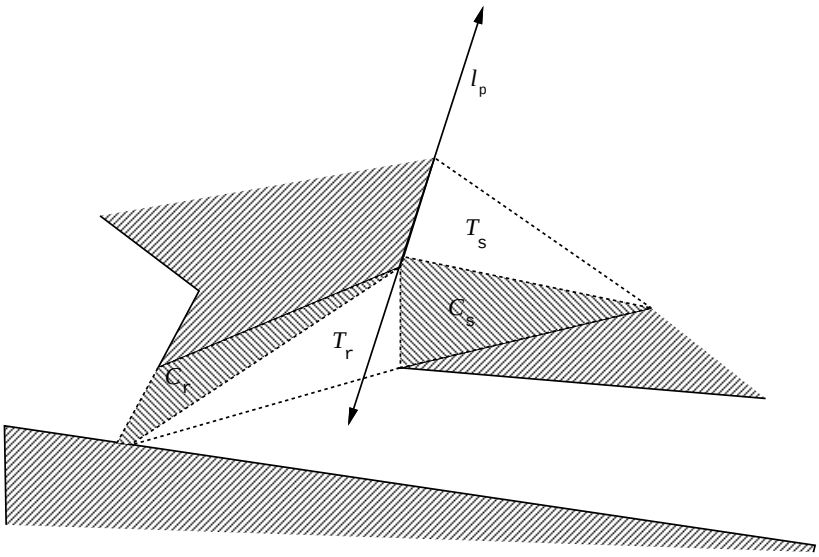

Figure 23: If l_p intersects t_2t_3 then case 2.1.2 applies.

Instance 3 *Vertex t_1 is moved.*

Vertex t_1 can move if the construction associated with case 2.3 is used to produce C_s with the result that Q_r becomes nonempty in P (see figure 24). We prevent this behavior by modifying case 2.3 to include the condition that the extended side f' may not be the side of any channel C_r supporting an associated triangle T_r. This restriction does not preclude the existence of an applicable case. An empty convex quadrilateral Q_s must exist or else case 2.3 would not apply. In Q_s, replace the point k where the extension from t_1 hits e with the other endpoint h'_s of e. Let the new quadrilateral be Q'_s. If Q'_s is empty and convex, then case 2.2 applies. If Q'_s is nonempty, then vertex x exists such that either h' or p' next moves to x. If Q_s is nonconvex and empty, then p' moves to t_1. At this point, extend an edge from t_1 again, but use the other incident edge. A new vertex t' is created next to t_1 in the direction of t_2. Let g'_s be the point where the extension hits e. Either quadrilateral h'_s, g'_s, t_1, t_2 is convex and empty, or a vertex v_s exists such that h_s, g_s, t_1, v_s is convex and empty. Thus case 2.3 holds using this different edge extension. If t_2 is the vertex chosen for Q'_s, then $Q'_r = t', h_r, p_r, g'_s$ is convex and empty in P The corresponding triangles T_r and T_s are identical and exist in P^s. If some other vertex v_s is chosen for Q'_s, then $Q'_r = t', h_r, p_r, t_3$ is convex and empty in P, t' replaces t_1, and T_r exists.

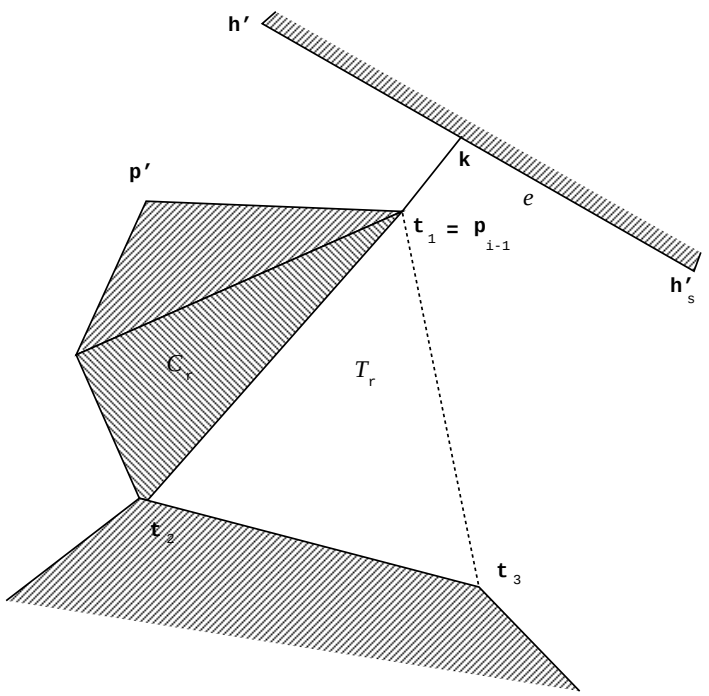

Figure 24: Extending edge t_2t_1 through t_1 makes Q_r nonempty in P.

Instance 4 *Vertex t_3 is cut off from T_r by the intended channel C_s.*

This instance resembles instance 2 where vertex t_3 moves onto edge t_2p_j. Channel C_s creates two new vertices on side t_2t_3 (or t_2p_j). Let these two vertices be q and q'. If q is closer to t_2, then q replaces t_3, $Q'_r = t_1, h_r, p_r, q$ is convex and empty in P, and T_r exists in P^s (see figure 25). If q' is the vertex closer to t_2, then T_r and T_s must share the area t_1, t_2, q', g_s (see figure 26). T_r becomes the triangle t_1, t_2, q' and T_s is chosen using vertices q', g_s and t_1 or T_r is chosen using t_1, t_2 and g_s and T_s becomes g_s, t_2, q'.$\diamond$

Instance 5 *Vertex t_2 is cut off from T_r by the intended channel C_s.*

This instance occurs only if t_2t_3 is interior to P^r. We attempted to construct channel C_s so that its ϵ-side lies along edge t_1t_2, but then T_r would no longer exists within P^s. A new vertex must be chosen to replace this t_2. Let the two vertices of C_s on edge t_1t_2 be q and q'. Whichever of q or q' lies on the same side of C_s as t_1 and t_3 will replace t_2 in T_r. T_r in its modified form still has one side along the side of channel C_r and its three vertices, if superimposed on P_r (and P), all see the entire

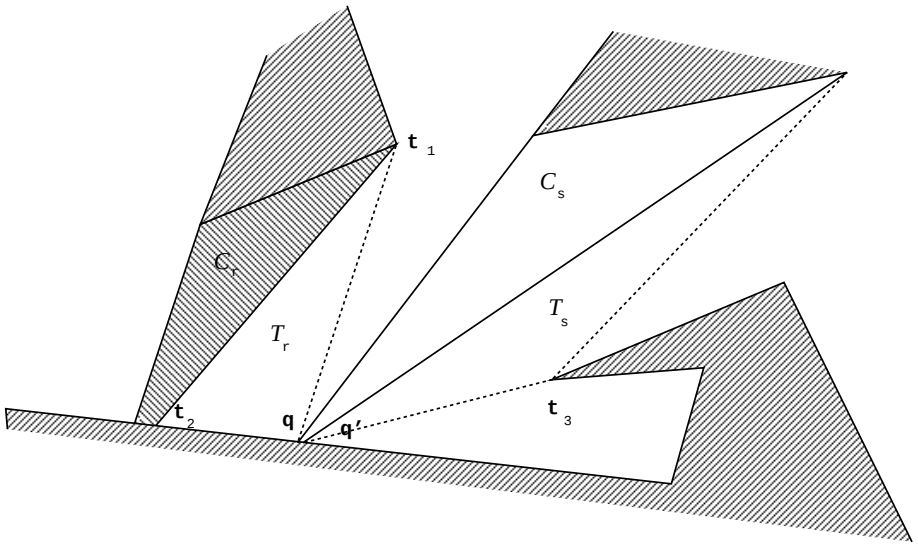

Figure 25: T_r shrinks to become triangle t_1, t_2, p.

area covered by C_r. Specifically, if q is closer to t_1, T_r becomes the triangle t_1, q, t_3 (see figure 27). The vertex v_s which completes Q_s lies on the other side of C_s, and triangles T_r and T_s now do not interfere with each other. If q' is closer to t_1, T_r shrinks to t_1, q', t_3 (see figure 28). In this case triangles T_r and T_s lie on the same side of C_s. By the choice of t_3 we know that vertex v_s which becomes the apex of T_s either is identical with the vertex t_3 or lies inside the quadrilateral g_s, h_s, q, t_3. Thus triangles T_r and T_s do not interfere with each other and both exist in P^s.$\diamond$

Instance 6 *Vertex t_1 is cut off from T_r by the intended channel C_s.*

In this case, we wish to place the ϵ-side of C_s along t_1t_2 by using an extension from some vertex h_s on a hole. When C_r was constructed, the additional condition from instance 2 above must have held or C_r would not have been constructed. We thus know that either there exists some other vertex t that is encountered before h_k when t_1t_2 rotates towards the extension through t_1 (if t_2t_3 is interior to P, then $t = t_3$), or the extension of the edge counterclockwise from t_1 intersects t_2h_s. In the first case, we first replace t_3 by t in T_r. If this replacement forces C_s to cut off vertex t_2 instead of t_1 (see figure 29), we continue the modifications as in instance 5. If the replacement does not result in instance 5, we return to our original t_3 and let q and q' be the two new vertices on t_1t_2. With q' closer to t_1, T_r shrinks to the triangle q, t_2, t_3

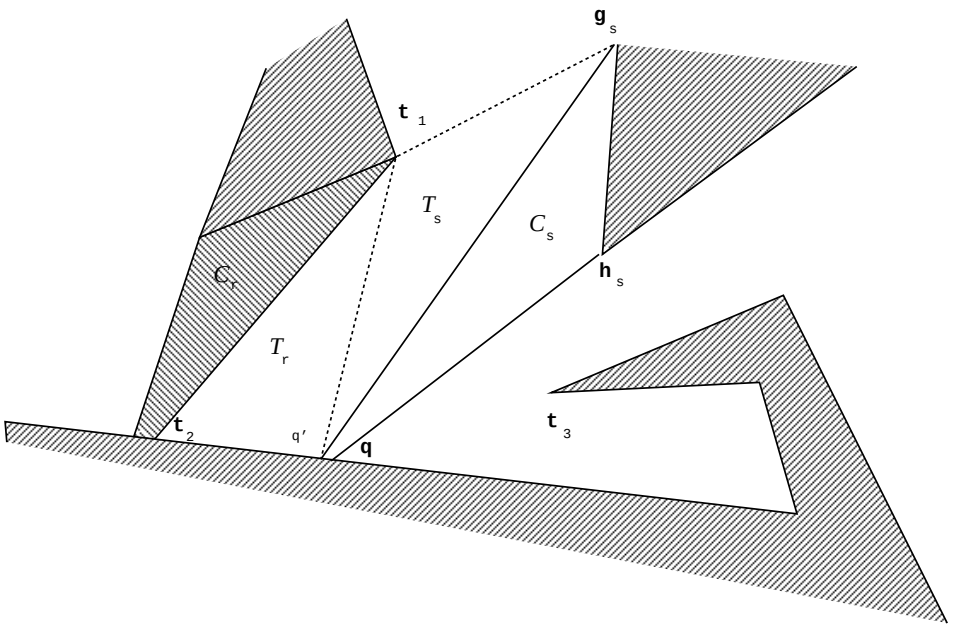

Figure 26: T_r shrinks to triangle t_1, t_2, q'.

(see figure 30) and T_s is chosen normally using q', t_1 and g_s. The modified T_r exists in P^s, and, since the vertex replacing t_1 lies on the side t_1t_2, every vertex of T_r still sees all of the area covered by C_r in P. If q is closer to t_1, T_r becomes the triangle q', t_2, t_3 (see figure 31). To make sure that T_r and T_s both exist, we look at the convex quadrilateral $Q_s = g_s, h_s, q', t_3$. If Q_s is empty, then T_s is the triangle g_s, q', t_3; if Q_s is nonempty, then one of the vertices in Q_s becomes the third vertex of T_s. In either case both T_r and T_s exist in P^s. Finally, if the edge counterclockwise from t_1 through t_1 intersects t_2h_s, we know that unless vertex h_s is the vertex where we actually start the search for a channel, the case 2.2 would have applied at an earlier step just as in instance 2. Consequently, $h_s = a$ and we attempt to apply the construction of case 2.1 to create channel C_s. If l_s and an extension of t_2t_3 through t_3 intersect and the area in between is empty, we move p' to t_2 and continue the search. Otherwise, we simply go ahead and construct the intended channel as above with q closer to t_1. $\diamond$

As demonstrated above, a triangle T_r may go through several modifications before all channels have been constructed (see figure 32), but the revised versions of cases 2.1, 2.2, and 2.3 guarantee that a satisfactory T_r will exist in P^h:

Case 3.1 *There exists a reflex vertex h_k on H with incident edges $e = h_kh_l$ and e'*

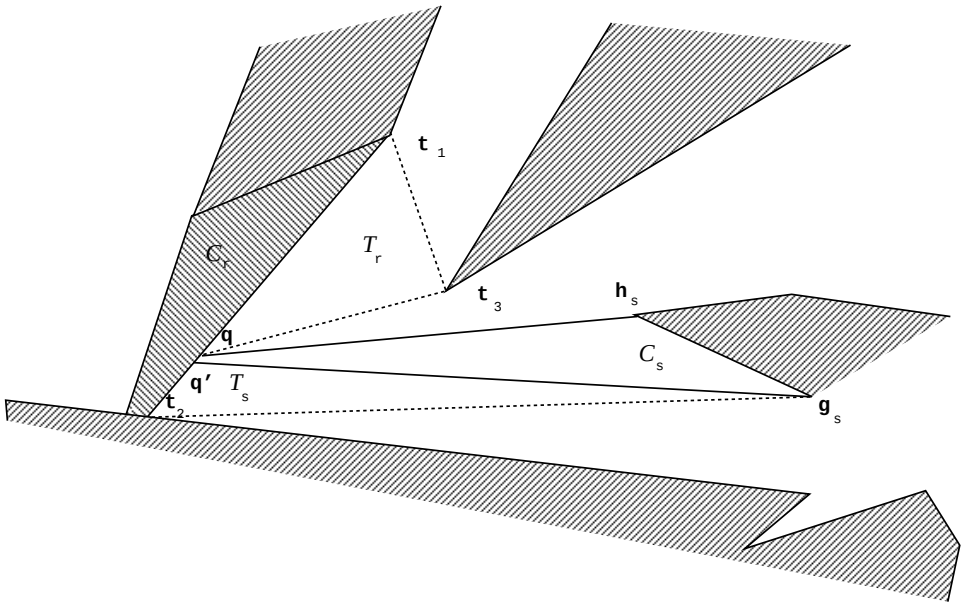

Figure 27: T_r shrinks to t_1, p, t_3.

and an edge $f = p_i p_j$ on B, such that the following conditions hold:

- *The extension of e' through h_k hits f at $q \neq p_j$.*
- *A half-plane defined by $\overline{h_k q}$ contains both e and p_j.*
- *There exists a vertex v on B or H and a quadrilateral $Q = h_l, h_k, q, v$ such that*
  1. *$v \notin \overline{q h_l}$*
  2. *v is the first vertex encountered by $h_l q$ as q slides along f*
  3. *$v = p_j \Rightarrow P \cup triangle(h_l, v, z) \neq \{h_l, v, z\}$*
  4. *Q is convex*
  5. *Q has an empty interior*
- *f is not the base of a triangle T with h_k designated t_3.*
- *There does not exist vertex u on H such that*
  1. *Excluding p_j, u is first vertex encountered by $h_l q$ as q slides along a line through f.*
  2. *The extension of an edge incident with u intersects $h_l q$.*
  3. *u, h_l, h_m is a left turn, where h_m is endpoint of edge $e'' \neq e$ incident with h_l.*

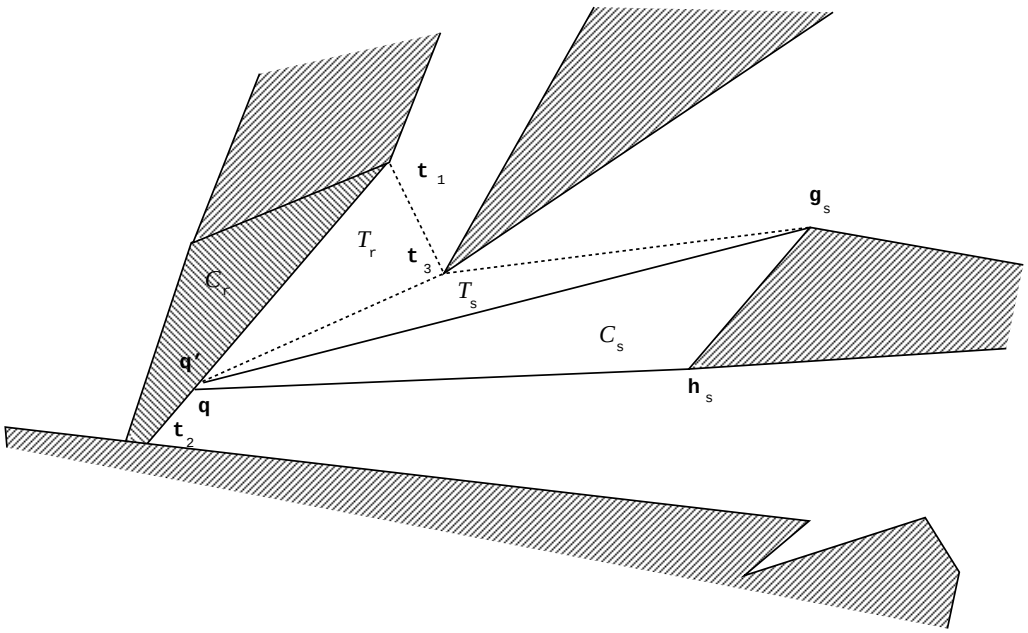

Figure 28: T_r shrinks to t_1, p', t_3.

Case 3.2 *There exists an edge $e = h_k h_l$ on H and an edge $f = p_i p_j$ on B such that the following conditions hold:*

- *A half-plane defined by $\overline{h_k p_i}$ contains both e and p_j.*
- *There exists a vertex v on H or B and a quadrilateral $Q = h_l, h_k, p_i, v$ such that*
  1. *$v \notin \overline{p_i h_l}$*
  2. *v is the first vertex encountered by $h_l p_i$ as p_i slides along f*
  3. *$v = p_j \Rightarrow P \cup triangle(h_l, v, z) \neq \{h_l, v, z\}$*
  4. *Q is convex*
  5. *Q has an empty interior*
- *There does not exist vertex u on H such that*
  1. *Excluding p_j, u is the first vertex encountered by $h_l p_i$ as p_i slides along a line through f.*
  2. *The extension of an edge incident to u intersects $h_l p_i$.*
  3. *u, h_l, h_m is a left turn, where h_m is endpoint of edge $e'' \neq e$ incident with h_l.*

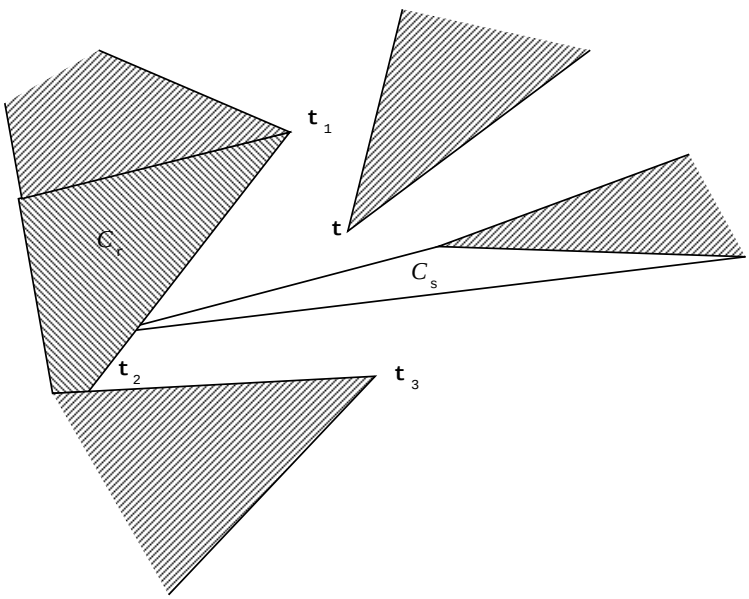

Figure 29: When t replaces t_3 in T_r, channel C_s cuts off t_2 instead of t_1 from T_r.

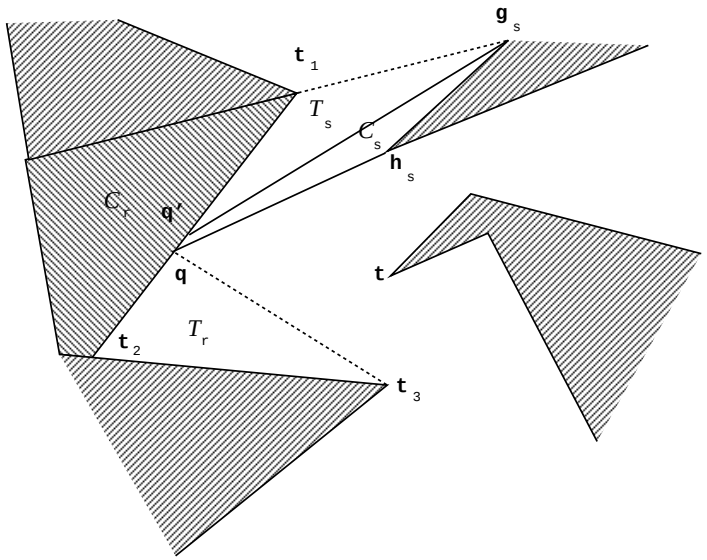

Figure 30: T_r shrinks to q, t_2, t_3.

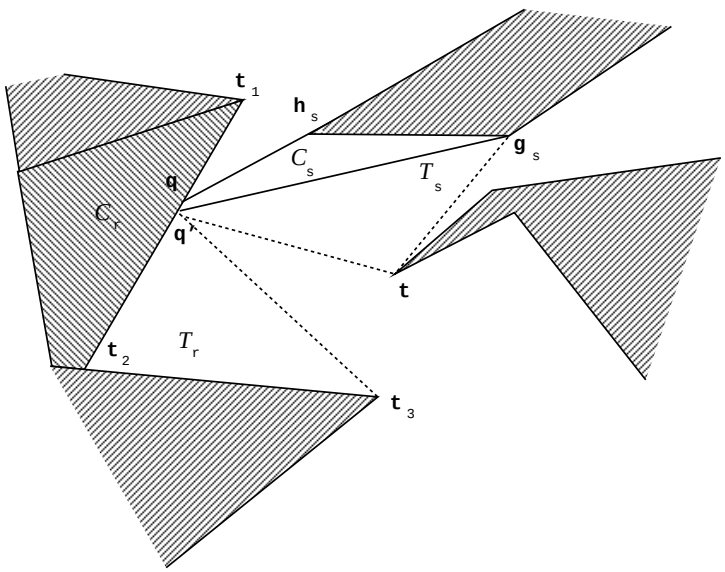

Figure 31: T_r shrinks to p', t_2, t_3.

Case 3.3 *There exists an edge $e = h_k h_l$ on H and a reflex vertex p_i with incident edges f anf f' on B, such that the following conditions hold:*

- *The extension of f' through p_i hits e at $k \neq h_l$.*
- *A half-plane defined by $\overline{p_i k}$ contains both f and h_l.*
- *There exists a vertex v on H or B and a quadrilateral $Q = h_l, k, p_i, v$ such that*
  1. *$v \notin \overline{p_i h_l}$*
  2. *v is the first vertex encountered by $h_l p_i$ as p_i slides along f*
  3. *$v = p_j \Rightarrow P \cup triangle(h_l, v, z) \neq \{h_l, v, z\}$*
  4. *Q is convex*
  5. *Q has an empty interior*
- *f' is not the base of a triangle with p_i designated t_1.*
- *There does not exist vertex u on H such that*
  1. *Excluding p_j, u is the first vertex encountered by $h_l p_i$ as p_i slides along a line through f.*
  2. *The extension of an edge incident to u intersects $h_l p_i$.*
  3. *$u, h_l h_m$ is a left turn, where h_m is endpoint of edge $e'' \neq e$ incident with h_l.*

When h channels have been constructed, every part of each channel is visible from a triangle and the interiors of these triangles are all disjoint. In addition, the construction of the channels results in a simple polygon P^h with at most $n + h$ vertices and no holes. The triangles $T_1, ..., T_h$ are inside this polygon. We remove the triangles and triangulate the remaining parts. Putting the pieces together again we have a triangulated polygon in which, after 3-coloring, we can place $\lfloor (n+h)/3 \rfloor$ vertex guards that see all of P^h (see figure 33). Since each of the triangles $T_1, ..., T_h$ has a guard at at least one vertex, these guards see all of the original polygon P. The guards are point guards, since P^h has vertices not present in P. This proves the following theorem:

Theorem 3.1 *$\lfloor (n+h)/3 \rfloor$ point guards are sufficient to cover the interior of a simple polygon of n vertices and with h holes.*

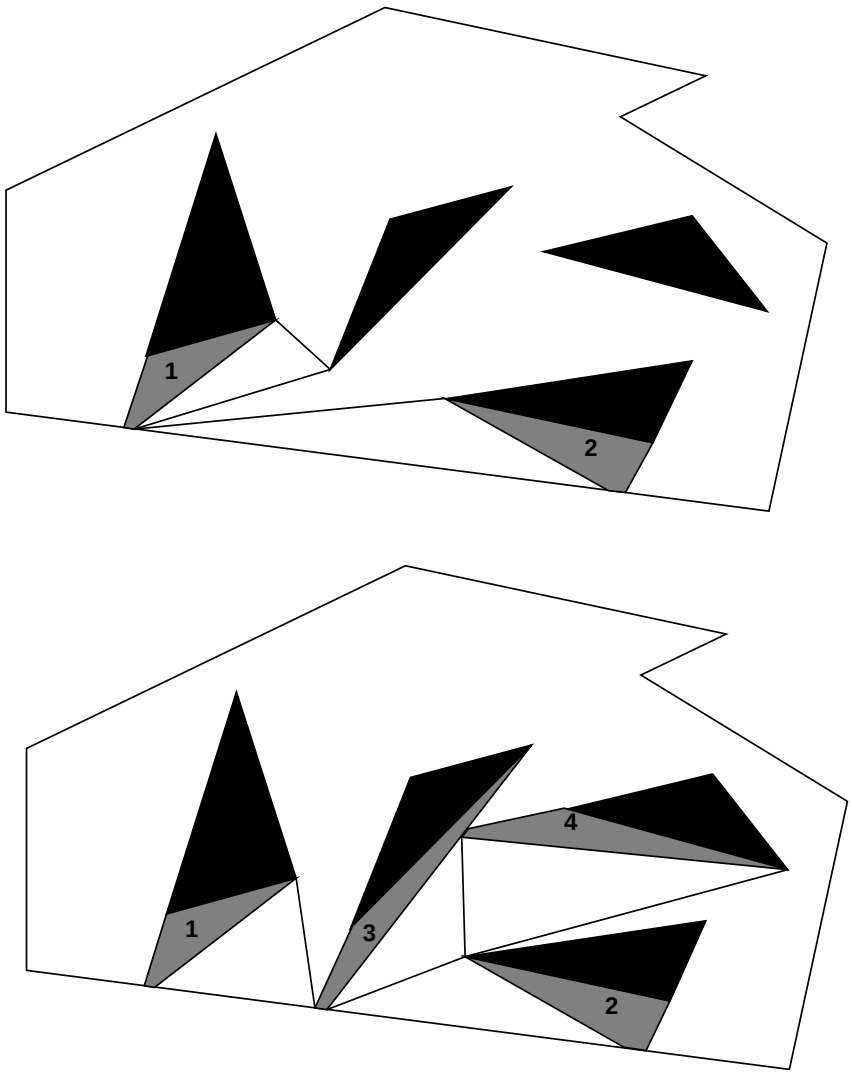

Figure 32: The channels (light color) are constructed in numerical order.

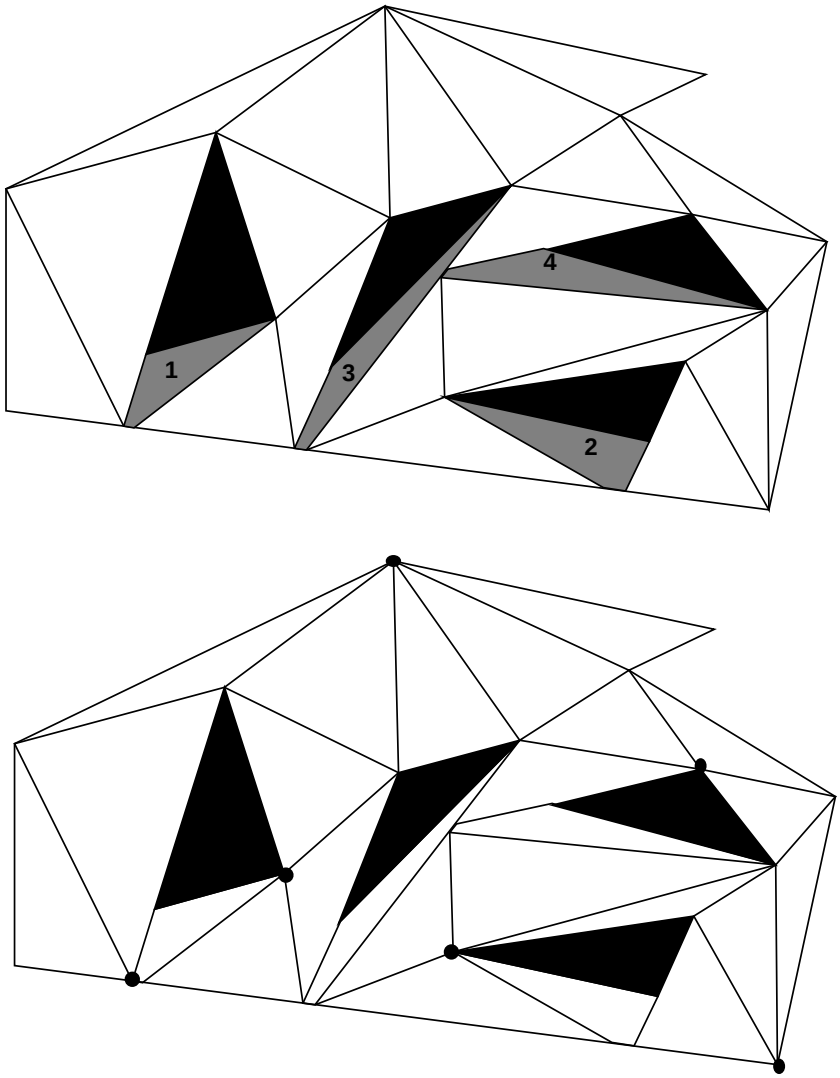

Figure 33: The hole-free polygon is triangulated and guards placed based on 3-coloring. These guards see all of the original polygon.

4 Algorithm for placing the guards

We use the following argument to show that $O(n^2)$ time will suffice to turn the polygon P into a simple polygon P^h with no holes and $n+h$ vertices: for each pair of vertices h' and p', $O(n)$ time suffices to check whether a channel between the edges e and f is possible; during the algorithm, at most $O(n)$ edge pairs are checked. Although the triangulation, 3-coloring, and placing of the guards within P^h can be done in linear time, (see section 1.2.1), we use the practical algorithm by Avis and Toussaint, which runs in $O(n \log n)$ time. Thus the overall time bound is $O(n^2)$.

Consider the case of one hole. Let H be the hole and B the exterior boundary of the polygon. Following the constructive proof of the theorem in section 2, we start by finding a vertex a on H, to serve as h'. We next find the edge first intersected by the extension of the edge clockwise from a through a and compute $r = p'$. Given the segment $h'p'$, we try each channel construction in turn, if none is possible either h' or p' is moved one step in a counterclockwise direction and we try again. All variable names are those used in sections 2 and 3.

Algorithm:(to turn P into P^1)
$h' \leftarrow$ leftmost vertex of H
$r \leftarrow p' \leftarrow$ intersection point between B and extension of edge clockwise from h' through h'
$p_i \leftarrow$ first vertex of B in clockwise direction from p
$q \leftarrow p'$
construct-a-channel

construct-a-channel:
while no channel is found **do**
    $x \leftarrow \emptyset$
    $next \leftarrow \emptyset$
    if $q = p'$ **then** try forward case 3.1
        if the case applies **then** a channel is found
        elseif Q is nonempty or nonconvex **then**
            $x \leftarrow$ first vertex encountered by rotating $h'p'$
        elseif vertex u exists **then** $x \leftarrow u$

else $x \leftarrow p_{i-1}$

if no success and $p' \neq$ r **then** try forward case 3.2

if the case applies **then** a channel is found

elseif Q is nonempty or nonconvex **then**

$x \leftarrow$ first vertex encountered by rotating $h'p'$

elseif vertex u exists **then** $x \leftarrow u$

if no channel is found and $p' = r$ **then** try forward case 3.1

if Q is nonempty or nonconvex **then**

$x \leftarrow$ first vertex encountered by rotating $h'p'$

elseif vertex u exists **then** $x \leftarrow u$

else $x \leftarrow p_{i-1}$

if no channel is found **and** $x = h_l$ **then** try backward case 3.1

if no channel is found **and** $x = p_{i-1}$ **then** try backward case 3.2

if no channel is found **then** try forward case 3.3 from p_i

if no channel is found **then** try case 3.3 from p_{i-1}

if no channel is found **then**

if $x \in H$ **then** $h' \leftarrow x$

else $p' \leftarrow x$

endwhile

Given a tentative pair h' and p', it takes linear time to try one construction. This is so, since all that is involved in an attempted construction is finding the intersections between two lines and checking whether any vertex lies inside the intended channel or the associated triangle. This can be done in linear time. Since only a constant number of constructions are tried, the while loop in the construct-a-channel algorithm runs in linear time. Once replaced, a particular vertex never again becomes the current h' or p'. Thus each time through the while loop either h' or p' must be previously unseen. Therefore, we run through the while loop at most n times and total time taken by construct-a-channel is $O(n^2)$. Picking an initial h' and finding the corresponding p' and p_i is also done in linear time. Thus the complete algorithm for one hole takes $O(n^2)$ time.

When there is more than one hole, the search for channels does not stop until all

holes are connected to the exterior. Keeping P and $H_1, ..., H_h$ as doubly linked lists enables us to update the polygon P in constant time, since when we are considering a particular pair h' and p', we are at the correct point in the lists to adjust pointers and delete and add appropriate vertices. Each time a channel is constructed, the search for the next channel continues in such a way that all but one of the vertex pairs producing a candidate channel contains a previously unseen vertex. To ensure this phenomenon, throughout the search, we record each place where the search jumps from one hole to another, by pushing the pair old h', old p' on a stack S. After a channel has been constructed, pop the stack and check if the indicated hole has been connected. If so, pop the stack again; else continue the search from the popped vertices. The vertex which last time became the new h' now lies on the exterior boundary and becomes the new p'. The search continues from there along a new route.

Algorithm: (to turn P into P^h)
$S \leftarrow \emptyset$
$j \leftarrow 0$
while $j < h$ **do**
    if $S = \emptyset$ **then**
        $h' \leftarrow a \leftarrow$ leftmost vertex on H
        $p' \leftarrow r \leftarrow$ intersection point between B and extension of e' through h'
        $p_i \leftarrow$ left endpoint of edge on which p' lies
        construct-a-channel
        $j \leftarrow j + 1$
    else
        $h' \leftarrow \text{pop}(S)$
        if $h' \in B$ **then**
            $\text{pop}(S)$
        else
            $q \leftarrow$ intersection point between B and extension of e' through h'
            $p' \leftarrow \text{pop}(S)$
            construct-a-channel
            $j \leftarrow j + 1$
        endif

endif

endwhile

Either at the start of the search or at any intermediary point when the stack is empty and holes still exist, we pick the leftmost hole vertex and the point where an extension of the edge clockwise from this vertex intersects the boundary as the place to continue the search. When the stack is empty, all vertices previously considered now lie on the current exterior boundary. Thus this initial pair will contain a previously unseen vertex. If no channel construction is found for this pair and p' is replaced by an obstructing vertex, this vertex may have been previously seen, but this can occur at most once each time we restart the search, i.e. at most once for each hole. In summary since the number of holes is h, we investigate at most $O(n+h)$ edge pairs for possible channel constructions. Spending $O(n)$ time on each edge pair gives us a total time of $O(n^2)$.

5 Conclusions

In this paper we studied the problem of assigning stationary guards in polygons with holes. We showed a bound for the number of guards required and gave an algorithm for placing the guards that runs in $O(n^2)$ time, where n is the total number of vertices. The algorithm was derived from the constructive proof of the upper bound.

We note that a new polygon P^h without holes was constructed from the original polygon P in such a way that several new vertices were created. Therefore, although the chosen guards are vertex guards in P^h, these vertices need not exist in P and the same guards are instead point guards in the original polygon. The problem of finding a tight bound on the number of vertex guards required to guard a polygon with h holes and a total of n vertices remains open. $\lfloor (n+h)/3 \rfloor$ vertex guards are certainly necessary but the best known upper bound is $\lfloor (n+2h)/3 \rfloor$ established by O'Rourke [8] in 1982.

6 Acknowledgements

We thank Joe O'Rourke for the many helpful comments and suggestions to improve the presentation of this paper.

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